

**ALGEBRAIC COMBINATORICS LEARNING SEMINAR FALL 2026**  
**LECTURE 5:**  
**INTRODUCTION TO NAKAJIMA QUIVER VARIETIES IN TYPE A**

RAJ GANDHI

CONTENTS

|                                           |   |
|-------------------------------------------|---|
| 1. Quiver representations                 | 1 |
| 1.1. Preliminary definitions and examples | 1 |
| 1.2. Indecomposable representations       | 2 |
| 2. Nakajima quiver varieties              | 3 |
| 2.1. Pre-quiver varieties                 | 3 |
| 2.2. Nakajima quiver varieties            | 5 |
| References                                | 8 |

We will follow [Kir16].

1. QUIVER REPRESENTATIONS

1.1. Preliminary definitions and examples.

1.1.1. *Quiver representations.* Let  $Q := (Q_0, Q_1)$  be a **quiver**, i.e,  $Q$  is a directed graph with finite vertex set  $Q_0 := \{q_1, \dots, q_n\}$  and finite edge set  $Q_1$ . A **representation**  $(V, x)$  of  $Q$  is an assignment of a vector space  $V_i$  to each vertex  $q_i$  and an assignment of a linear transformation  $x_h: V_i \rightarrow V_j$  whenever there is an edge  $h: q_i \rightarrow q_j$  in  $Q_1$ .

Note that a representation for  $Q$  can be viewed as an element of  $\text{Hom}(V) := \bigoplus_{(h: i \rightarrow j) \in Q_1} \text{Hom}(V_i, V_j)$ . Two quiver representations will be considered **equivalent** up to an action of  $\text{GL}(V) := \bigoplus_{q_i \in Q_0} \text{GL}(V_i)$ , where  $g := (g_i)_{q_i \in Q_0}$  acts on  $f := (f_h)_{h \in Q_1}$  by

$$g \cdot f := (g_j \circ f_h \circ g_i^{-1})_{(h: i \rightarrow j) \in Q_1}.$$

This essentially means that quiver representations are the same "up to basis", or perhaps can be thought of as equivalent "up to row spans and column spans" of the matrix representatives. The right action of  $g_i^{-1}$  means the basis of  $V_i$  can be changed, but the span of the image of  $f \circ g_i^{-1}$  is the same as that of  $f$ , while the left action of  $g_j$  on  $f \circ g_i^{-1}$  means that the basis of  $W_j$  can be changed, so that any subspace of  $W$  with the same dimension as the image of  $f \circ g_i^{-1}$  is considered equivalent to the image of  $f \circ g_i^{-1}$ .

1.1.2. *Morphisms of quiver representations.* A **morphism** of representations  $f: (V, x) \rightarrow (W, y)$  of  $Q$  is a collection of linear transformations  $\{f_i: V_i \rightarrow W_i\}_{q_i \in Q_0}$  such that, for every edge  $h: q_i \rightarrow q_j$  in  $Q_1$ , the diagram below commutes:

$$\begin{array}{ccc} V_i & \xrightarrow{f_i} & W_i \\ x_h \downarrow & & \downarrow y_h \\ V_j & \xrightarrow{f_j} & W_j \end{array}$$

We say that  $f$  is an **isomorphism** if every  $f_i$  is a vector space isomorphism.

**Example 1.1.** Consider the following directed  $A_2$  quiver:

$$\bullet \longrightarrow \bullet$$

Two examples of quiver representations for this quiver are illustrated below:

$$\mathbb{C}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \mathbb{C}^2 \qquad \mathbb{C}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} \mathbb{C}^2$$

Observe that these two quiver representations are *not* isomorphic. If they were, then there would be isomorphisms  $f_1, f_2: \mathbb{C}^2 \xrightarrow{\sim} \mathbb{C}^2$  such that the following diagram would commute:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} & \mathbb{C}^2 \\ \downarrow f_1 & & \downarrow f_2 \\ \mathbb{C}^2 & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} & \mathbb{C}^2 \end{array}$$

This would then imply that

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \circ f_1 &= f_2 \circ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \implies \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} &= f_2 \circ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \circ f_1^{-1}. \end{aligned}$$

However, the left side has rank 1 while the right side has rank 2, which is not possible.

Put, differently, the matrices  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$  are not in the same  $GL(\mathbb{C}^2) \times GL(\mathbb{C}^2)$  orbit.

## 1.2. Indecomposable representations.

**1.2.1. Direct sums.** Let  $(V, x)$  and  $(W, y)$  be two representations for  $Q = (Q_0, Q_1)$ . The **direct sum** of these two representations is the representation  $(V, x) \oplus (W, y) := (V \oplus W, x \oplus y)$ , by which we mean the representation with the vector space  $V_i \oplus W_i$  at the vertex  $q_i$  and the linear transformation  $x_h \oplus y_h: V_i \oplus W_i \rightarrow V_j \oplus W_j$  at the edge  $h: i \rightarrow j$  of  $Q_1$ .

**Example 1.2.** Consider the following directed  $A_3$  quiver:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

Two examples of quiver representations for this quiver are illustrated below

$$\mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{1} \mathbb{C} \qquad \mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{0} 0$$

The direct sum of these two quiver representations is

$$\mathbb{C}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \mathbb{C}^2 \xleftarrow{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \mathbb{C} \oplus 0$$

**1.2.2. Indecomposables and Krull–Schmidt.** A quiver representation is **indecomposable** if it is not isomorphic to the direct sum of two non-trivial quiver representations

**Theorem 1.3.** (Krull–Schmidt) *Any finite-dimensional representation of a quiver is a direct sum of indecomposable representations, and the decomposition as a direct sum is unique up to reordering and isomorphism of the summands.*

**Remark 1.4.** In the  $A_1$  case, Theorem 1.3 just says that vector spaces have bases and are determined by their dimension.

Consider the  $A_n$  Dynkin diagram, which has  $n$  vertices:

$$\bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \dots \text{ --- } \bullet \text{ --- } \bullet$$

If the graph underlying  $Q$  is the  $A_n$  Dynkin diagram, then the indecomposable representations of  $Q$  all have the following form:

$$g_{i,i+1,\dots,j}: 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow \mathbb{C} \xrightarrow{\text{id}_{\mathbb{C}}} \dots \xrightarrow{\text{id}_{\mathbb{C}}} \mathbb{C} \longrightarrow 0 \longrightarrow \dots \longrightarrow 0$$

In other words, the indecomposables  $(V, x)$  satisfy

$$V_k = \begin{cases} \mathbb{C}, & \text{if } k = i, \dots, j \text{ for } i \leq j \\ 0, & \text{otherwise.} \end{cases}$$

Observe that the number of indecomposables equals the number of positive roots for the  $A_n$  quiver.

1.2.3. *Rephrasing rank-nullity for type  $A_2$ .* Any linear transformation of vector spaces  $f: V \rightarrow W$  can be viewed as a representation of the  $A_2$  quiver. Recall the **rank-nullity theorem**, that there is a vector space isomorphism:

$$V \simeq \ker(f) \oplus \operatorname{im}(f).$$

The morphism  $f_1: \ker(f) \rightarrow 0$  is itself a direct sum of morphisms of the form

$$\mathbb{C} \longrightarrow 0$$

and the morphism  $f_2: \operatorname{im}(f) \rightarrow \operatorname{im}(f)$  is itself a direct sum of morphisms of the form

$$\mathbb{C} \longrightarrow \mathbb{C}$$

If  $f$  is surjective, then  $f$  decomposes as  $f \simeq f_1 \oplus f_2$ . If  $f$  is *not* surjective, then we consider a third type of indecomposable:

$$f_3: (0 \longrightarrow \mathbb{C})^{\circ(\dim(W) - \operatorname{rank}(f))}$$

So, when  $f$  is not surjective, it decomposes as  $f \simeq f_1 \oplus f_2 \oplus f_3$ .

In particular, specifically for the  $A_2$  quiver, Gabriel's theorem is equivalent to the rank-nullity theorem.

## 2. NAKAJIMA QUIVER VARIETIES

### 2.1. Pre-quiver varieties.

2.1.1. *Subrepresentations and kernels.* We need two important types of quiver representations that can be formed from a given one.

Let  $Q = (Q_0, Q_1)$  be a quiver, with vertices  $Q_0 = \{q_1, \dots, q_n\}$ , and  $(V, x)$  a representation of  $Q$ . A **subrepresentation** of  $(V, x)$  is a representation  $(W, y)$  of  $Q$  such that  $W_i$  is a vector subspace of  $V_i$  for each  $i$ , and such that the following diagram commutes for every arrow  $(h: i \rightarrow j) \in Q_1$ :

$$\begin{array}{ccc} V_i & \xrightarrow{x_h} & V_j \\ \uparrow & & \uparrow \\ W_i & \xrightarrow{y_h} & W_j \end{array}$$

That is to say,  $y_h = (x_h)|_{W_i}$  for every arrow  $h: i \rightarrow j$  in  $Q_1$ .

One important example of a subrepresentation of  $(V, x)$  is the **kernel**  $(\ker(f), x|_{\ker(f)})$  of a morphism of quiver representations  $f: (V, x) \rightarrow (W, y)$ . In the kernel subrepresentation, we have  $(\ker(f))_i = \ker(f_i)$ , and for every arrow  $(h: i \rightarrow j) \in Q_1$ , we  $(x|_{\ker(f)})_h = (x_h)|_{\ker(f_i)}$ .

2.1.2. *Framed representations.* A **dimension vector** is a sequence of non-negative integers  $(d_1, \dots, d_n)$ . The **dimension vector** of a sequence of vector spaces  $W := (W_1, \dots, W_n)$  (or of a representation  $(W, y)$  of  $Q$ ) is the sequence  $\dim(W) := (\dim(W_1), \dots, \dim(W_n))$ .

To construct a  $\dim(W)$ -framed representation of  $(V, x)$ , first construct the quiver  $Q' = (Q'_0, Q'_1)$  by doubling the vertices of  $Q$ , i.e.,  $Q'_0 = (q_1, q'_1, \dots, q_n, q'_n)$ , and for each  $i$ , declare an arrow  $q_i \rightarrow q'_i$ . A  **$\dim(W)$ -framed representation** of  $(V, x)$  is a quiver representation  $(V', x')$  of  $Q'$ , where  $(V', x')$  restricted to the sub-quiver  $Q$  has representation  $(V, x)$ , and such that a vector space  $V'_i$  of dimension  $(\dim(W))_i$  is placed at  $q'_i$ , and such that there is a morphism  $j_i := x'|_{q_i \rightarrow q'_i}: V_i \rightarrow V'_i$  for every  $i$ .

The original vertices  $(q_i)$  are called **gauged vertices** and the new vertices  $(q'_i)$  are called **framed vertices**. We will often put squares around the framed vertices of a quiver, to distinguish them from gauged vertices.

**Example 2.1.** Consider the  $A_2$  quiver:

$$\bullet \xrightarrow{h} \bullet$$

A  $(w_1, w_2)$ -framed representation of the representation  $(V, x)$  below

$$\mathbb{C}^{v_1} \xrightarrow{x_h} \mathbb{C}^{v_2}$$

is any representation of  $(A_2)'$  given by

$$\begin{array}{ccc} \boxed{\mathbb{C}^{w_1}} & & \boxed{\mathbb{C}^{w_2}} \\ j_1 \uparrow & & j_2 \uparrow \\ \mathbb{C}^{v_1} & \xrightarrow{x_h} & \mathbb{C}^{v_2} \end{array}$$

Observe that the kernel of  $j = (j_1, j_2)$  is not necessarily a subrepresentation of  $(V, x)$ . For example, consider the representation  $(V, x)$  defined by the linear transformation  $x_h = \text{id}_{\mathbb{C}^2} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ . Suppose that  $j_1$  is the transformation

$$\mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad (x, y) \mapsto (x, 0)$$

and that  $j_2$  is the linear transformation

$$\mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad (x, y) \mapsto (0, y).$$

As  $(x_h)|_{\ker(j_1)}$  does not lie in  $\ker(j_2) = \{(x, 0) \mid x \in \mathbb{C}\}$ , the kernel of  $j$  does *not* define a subrepresentation of  $(V, x)$ .

A  $\dim(W)$ -**framed representation** of  $Q$  with dimension vector  $D$  is a  $\dim(W)$ -framed representation of a representation  $(V, x)$  of  $Q$ , such that  $\dim(V) = D$ .

In this section, we will always consider two  $\dim(W)$ -framed representations of  $Q$  **equivalent** up to an action of  $GL(V) := \bigoplus_{q_i \in Q_0} GL(V_i)$ , i.e., up to the action of  $GL(V)$  *only at gauged vertices*.

**2.1.3. The stability condition: pre-quiver varieties.** We are now ready to define the 1st approximation of the quiver variety. A **pre-quiver variety**  $\mathcal{M}(\dim(V), \dim(W), Q)$  is the set of  $\dim(W)$ -framed representations of  $Q$  with dimension vector  $\dim(V)$  that satisfy the following **stability condition**:

*If  $(V, x)$  is a  $\dim(W)$ -framed representation  $Q$  and  $V'$  is a subrepresentation of  $(V, x)$  contained in  $\ker(j)$ , then  $V' = (0)$ .*

Although people study other ‘stability conditions’ as well, we will stick to this one for simplicity.

**Remark 2.2.** The stability condition is used to carve out a “stable” open subset of

$$\bigoplus_{(h: i \rightarrow j) \in Q_1} \text{Hom}(V_i, V_j) \oplus \bigoplus_{q_i \in Q_0} \text{Hom}(V_i, V'_i),$$

such that the open subset modulo that action of  $GL(V)$  on gauged vertices is a quasi-projective algebraic variety.

**Example 2.3.** Consider the single  $A_1$  quiver  $Q$ :

$$\bullet$$

The quiver  $Q'$  is as follows:

$$\bullet \longrightarrow \boxed{\bullet}$$

Fix dimension vectors  $v := \dim(V)$  and  $w := \dim(W)$ , with  $w \geq v$ , and consider the pre-quiver variety  $\mathcal{M}(v, w, Q)$ . Thus, we can view  $\mathcal{M}(v, w, Q)$  as the quotient of the variety of linear transformations

$$(\text{Hom}(\mathbb{C}^v, \mathbb{C}^w) \setminus \{\text{unstable part}\}) / GL_v(\mathbb{C}).$$

Observe that every subspace of  $V$  is also a subrepresentation of  $V$ . Suppose  $j: \mathbb{C}^v \rightarrow \mathbb{C}^w$  is a linear transformation, and assume that  $V'$  is a subspace of  $\ker(j)$ . Then by the stability condition,  $V' = (0)$ . As  $\ker(j)$  is a subspace of itself, it therefore must too be  $(0)$ . So, the pre-quiver variety we are looking at is the set of all injective morphisms

$$j: \mathbb{C}^v \hookrightarrow \mathbb{C}^w$$

up to change of basis at the gauged vertex. This is precisely the Grassmannian  $\text{Gr}(v, w)$ . Thus,  $\mathcal{M}(v, w, Q) = \text{Gr}(v, w)$  in this case.

**Example 2.4.** Let us repeat this story for the arbitrary  $A_n$  quiver  $Q$ :

$$\bullet \xrightarrow{h_1} \bullet \xrightarrow{h_2} \bullet \xrightarrow{h_3} \dots \xrightarrow{h_{n-2}} \bullet \xrightarrow{h_{n-1}} \bullet$$

Fix dimension vectors  $\dim(V) = (v_1, \dots, v_n)$  and  $\dim(W) = (0, \dots, 0, w_n)$ , with  $v_1 < v_2 < \dots < v_n < w_n$ , and consider the pre-quiver variety  $\mathcal{M}(\dim(V), \dim(W), Q)$ . A  $\dim(W)$ -framed representation  $(V, \chi)$  of  $Q$  is one that looks like:

$$\begin{array}{ccccccc} \boxed{0} & & \boxed{0} & & \dots & & \boxed{0} & & \boxed{\mathbb{C}^{w_n}} \\ j_1 \uparrow & & j_2 \uparrow & & & & j_{n-1} \uparrow & & j_n \uparrow \\ \mathbb{C}^{v_1} & \xrightarrow{x_{h_1}} & \mathbb{C}^{v_2} & \xrightarrow{x_{h_2}} & \dots & \xrightarrow{x_{h_{n-2}}} & \mathbb{C}^{v_{n-1}} & \xrightarrow{x_{h_{n-1}}} & \mathbb{C}^{v_n} \end{array}$$

The kernel of  $j$  is  $(\mathbb{C}^{v_1}, \dots, \mathbb{C}^{v_{n-1}}, \ker(j_n))$ . Observe that one can construct the following subrepresentation of  $(V, \chi)$ , which lies in  $\ker(j)$ :

$$0 \longrightarrow \dots \longrightarrow 0 \longrightarrow \ker(x_{h_i}) \longrightarrow 0 \longrightarrow \dots \longrightarrow 0$$

If the stability condition is imposed, then we require  $\ker(x_{h_i}) = (0)$ , forcing the map  $x_{h_i}$  to be injective. In addition,  $\ker(j_n)$  defines the following subrepresentation of  $(V, \chi)$  which, by commutativity of the diagram, is contained in  $\ker(j)$ :

$$0 \longrightarrow \dots \longrightarrow 0 \longrightarrow \ker(j_n)$$

Thus,  $\ker(j_n) = 0$  as well, so the map  $j_n$  is injective. Thus,  $\mathcal{M}(\dim(V), \dim(W), Q)$  can be identified with the set of injective sequences of maps, up to changes of basis at gauged vertices:

$$\mathbb{C}^{v_1} \hookrightarrow \mathbb{C}^{v_2} \hookrightarrow \dots \hookrightarrow \mathbb{C}^{v_k} \hookrightarrow \mathbb{C}^{w_n}.$$

Thus  $\mathcal{M}(\dim(V), \dim(W), Q)$  is the partial flag variety  $\text{Fl}(v_1, v_2, \dots, v_n; \mathbb{C}^{w_n})$ .

**Remark 2.5.** In general, a pre-quiver variety for a type  $A_n$  quiver is a **type A pre-quiver variety**, and not all of these are isomorphic to partial flag varieties.

## 2.2. Nakajima quiver varieties.

**2.2.1. Doubled quivers.** If  $Q = (Q_0, Q_1)$  is a quiver, then the **doubled quiver**  $\overline{Q} = (Q_0, \overline{Q}_1)$  is the quiver with vertex set  $Q_0$  and edge set  $\overline{Q}_1 := \{h: i \rightarrow j \mid h \in Q_1\} \cup \{\overline{h}: j \rightarrow i \mid h \in Q_1\}$ , i.e.,  $\overline{Q}_1$  has all the edges in  $Q_1$ , plus a new edge with the opposite orientation for every edge in  $Q_1$ . If  $\overline{h} \in \overline{Q}_1 \setminus Q_1$ , then we define the notation  $\overline{(\overline{h})} := h \in Q_1$ .

**Example 2.6.** The doubled quiver corresponding to

$$\bullet \xrightarrow{h} \bullet$$

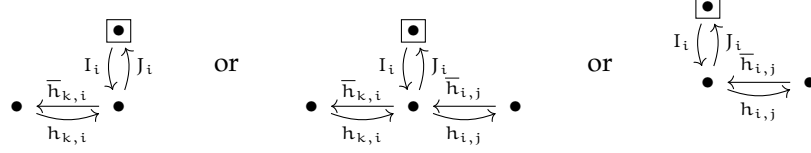
is the quiver

$$\bullet \begin{array}{c} \xleftarrow{\overline{h}} \\ \xrightarrow{h} \end{array} \bullet$$

Let  $Q = (Q_0, Q_1)$  be a quiver, and recall the quiver  $Q'$  with twice as many vertices as  $Q$ . We will consider the doubled quiver  $\overline{Q'}$ . A **doubled**  $\dim(W)$ -framed representation of  $Q$  with dimension vector  $\dim(V)$  is a representation of  $\overline{Q'}$  such that the vector space at the  $i$ -th framed vertex  $q'_i$  has dimension  $(\dim(W))_i$  and at the  $i$ -th gauged vertex  $q_i$  has dimension  $(\dim(V))_i$ .

**2.2.2. The moment map condition: Nakajima quiver varieties.** We will, at last, impose one final condition to obtain the *Nakajima quiver variety* from the pre-quiver variety. A **Nakajima quiver variety**  $\overline{\mathcal{M}}(\dim(V), \dim(W), Q)$  is the set of *doubled*  $\dim(W)$ -framed representations of  $Q$  with dimension vector  $\dim(V)$  that satisfy the following **moment map condition**, followed by the stability condition:

For every oriented subgraph of  $\overline{Q'}$  of the form (where  $h_{i,j}: q_i \rightarrow q_j$  and  $h_{k,i}: q_k \rightarrow q'_i$  and  $J_i: q_i \rightarrow q'_i$ )



we impose the condition  $\sum_{\bar{h} \in \overline{Q'_{1, \text{gauged}}}, \text{head}(\bar{h})=i} \epsilon(\bar{h}) x_{\bar{h}} x_h + I_i J_i = 0$ , where  $\epsilon(\bar{h}) = \begin{cases} 1, & \text{if } \bar{h} \in Q_1, \\ -1, & \text{otherwise.} \end{cases}$

**Remark 2.7.** Note that the trace map defines a non-degenerate bilinear form on  $\text{Hom}(V, W) \times \text{Hom}(W, V)$  (with  $V, W$  complex vector spaces), sending  $(f, g) \mapsto \text{tr}(g \circ f)$ , and hence  $\text{Hom}(W, V)$  can be identified with the dual of  $\text{Hom}(V, W)$  with respect to this bilinear form. Thus, the vector space  $\text{Hom}(V, W)$  has cotangent bundle

$$T^*\text{Hom}(V, W) \simeq \text{Hom}(V, W) \times \text{Hom}(V, W)^* \simeq \text{Hom}(V, W) \times \text{Hom}(W, V),$$

allowing one to identify the space of representations of a doubled quiver  $\overline{Q}$  with the cotangent bundle of the space of quiver representations of  $Q$ .

For the space of  $\dim(W)$ -framed representation of the set of  $\dim(V)$ -dimensional representations of a quiver  $Q$ , the action of  $\text{GL}(V)$  (i.e., the action at *gauged vertices*) on this space of these representations induces an action of  $\text{GL}(V)$  on the cotangent bundle of this space (which is the space of such doubled  $\dim(W)$ -framed representations). This induces a “moment map” from this cotangent bundle to  $\mathfrak{gl}(V)$ , whose zero level set  $\mu^{-1}(0)$  is precisely the subset of elements of this cotangent bundle satisfying the *moment map condition*. Next, imposing the stability condition on  $\mu^{-1}(0)$  cuts out a “stable” open subset, whose quotient by  $\text{GL}(V)$  is a “hyper-kähler variety”.

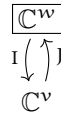
**Example 2.8.** Consider again the  $A_1$  quiver  $Q$ :



We will consider the doubled quiver  $\overline{Q'}$ :



Fix the dimension vectors  $v := \dim(V)$  and  $w := \dim(W)$  with  $v \leq w$ . Consider an arbitrary  $\dim(W)$ -framed quiver representation with these dimension vectors:



Imposing the moment map condition, we must have that

$$IJ = 0.$$

Setting  $X := JI$ , we see that  $X^2 = J(IJ)I = 0$ .

We can now impose the stability condition on the subset of the doubled  $w$ -framed representations satisfying the moment map condition, and by the earlier analysis, this will amount to the requirement that  $J$  is injective. Therefore,  $X(\mathbb{C}^w) \subseteq \mathbb{C}^v$  and  $X(J(\mathbb{C}^v)) = J(IJ)(\mathbb{C}^v) = 0$ , so the Nakajima quiver variety  $\overline{\mathcal{M}}(v, w, Q)$  can be viewed as the following set of injective maps, up to change of basis at the gauged vertex,

$$\{\mathbb{C}^v \hookrightarrow \mathbb{C}^w \mid X(\mathbb{C}^w) \subseteq \mathbb{C}^v \text{ and } X(\mathbb{C}^v) = 0 \text{ for some } X \in \text{End}(\mathbb{C}^w)\}$$

which is the Springer realization for the cotangent bundle of the Grassmannian  $T^*(\text{Gr}(v, w))^1$ .

**Example 2.9.** Let us again repeat this story for the arbitrary  $A_n$  quiver  $Q$ :

$$\bullet \xrightarrow{h_1} \bullet \xrightarrow{h_2} \bullet \xrightarrow{h_3} \dots \xrightarrow{h_{n-2}} \bullet \xrightarrow{h_{n-1}} \bullet$$

Fix dimension vectors  $\dim(V) = (v_1, \dots, v_n)$  and  $\dim(W) = (0, \dots, 0, w_n)$ , with  $v_1 < v_2 < \dots < v_n < w_n$ . A doubled  $\dim(W)$ -framed representation  $(V, x)$  of  $Q$  is one that looks like:

$$\begin{array}{ccccccc} \boxed{0} & \boxed{0} & & \dots & \boxed{0} & \boxed{\mathbb{C}^{w_n}} \\ \downarrow I_1 & \downarrow I_2 & & & \downarrow I_{n-1} & \downarrow I_n \\ \mathbb{C}^{v_1} & \xleftarrow{x_{h_1}} \mathbb{C}^{v_2} & \xleftarrow{x_{h_2}} \dots & \xleftarrow{x_{h_{n-2}}} \mathbb{C}^{v_{n-1}} & \xleftarrow{x_{h_{n-1}}} \mathbb{C}^{v_n} \\ \uparrow J_1 & \uparrow J_2 & & & \uparrow J_{n-1} & \uparrow J_n \end{array}$$

The moment map condition implies the following equations:

$$\begin{aligned} -x_{h_1} x_{h_1} &= 0; \\ -x_{h_{i+1}} x_{h_{i+1}} + x_{h_i} x_{h_i} &= 0, \quad i = 1, \dots, n-2; \\ x_{h_{n-1}} x_{h_{n-1}} + I_n J_n &= 0. \end{aligned}$$

Setting  $X := J_n I_n$ , we see that

$$\begin{aligned} X^2 &= J_n (I_n J_n) I_n = -J_n (x_{h_{n-1}} x_{h_{n-1}}) I_n \\ X^3 &= -J_n (x_{h_{n-1}} x_{h_{n-1}}) (I_n J_n) I_n = J_n (x_{h_{n-1}} x_{h_{n-1}}) (x_{h_{n-1}} x_{h_{n-1}}) I_n \\ &= J_n x_{h_{n-1}} (x_{h_{n-1}} x_{h_{n-1}}) x_{h_{n-1}} I_n \\ &= J_n x_{h_{n-1}} (x_{h_{n-2}} x_{h_{n-2}}) x_{h_{n-1}} I_n \end{aligned}$$

Continuing inductively, we observe that

$$X^i = (-1)^{i+1} J_n x_{h_{n-1}} \dots x_{h_{n-i+1}} x_{h_{n-i+1}} \dots x_{h_{n-1}} I_n.$$

Thus, at the  $i = n$  step, we see that

$$X^n = (-1)^n J_n x_{h_{n-1}} \dots x_{h_1} x_{h_1} \dots x_{h_{n-1}} I_n.$$

At the  $i = n+1$  step,  $X^{n+1}$  will contain a factor of the form  $x_{h_1} x_{h_1}$ , which is 0, and hence  $X^{n+1} = 0$ .

Now, impose the stability condition to observe (as before) that  $J_n$  and all the  $x_{h_i}$  maps are injective. Now observe that

$$\begin{aligned} X \circ J_n \circ x_{h_{n-1}} \circ \dots \circ x_{h_i} (\mathbb{C}^{v_i}) &= -J_n (I_n J_n) x_{h_{n-1}} \circ \dots \circ x_{h_i} (\mathbb{C}^{v_i}) \\ &= -J_n \circ x_{h_{n-1}} \circ (x_{h_{n-1}} \circ x_{h_{n-1}}) \circ x_{h_{n-2}} \circ \dots \circ x_{h_i} (\mathbb{C}^{v_i}) \\ &= -J_n \circ x_{h_{n-1}} \circ (x_{h_{n-2}} \circ x_{h_{n-2}}) \circ x_{h_{n-2}} \circ \dots \circ x_{h_i} (\mathbb{C}^{v_i}) \\ &\vdots \\ &= -J_n \circ x_{h_{n-1}} \circ x_{h_{n-2}} \circ \dots \circ (x_{h_i} \circ x_{h_i}) (\mathbb{C}^{v_i}) \\ &= -J_n \circ x_{h_{n-1}} \circ x_{h_{n-2}} \circ \dots \circ x_{h_{i-1}} \circ x_{h_{i-1}} (\mathbb{C}^{v_i}) \\ &\subseteq J_n \circ x_{h_{n-1}} \circ x_{h_{n-2}} \circ \dots \circ x_{h_{i-1}} (\mathbb{C}^{v_{i-1}}) \end{aligned}$$

Thus, we can view  $X(\mathbb{C}^{v_i}) \subseteq \mathbb{C}^{v_{i-1}}$ . Therefore, the Nakajima quiver variety  $\overline{\mathcal{M}}(\dim(V), \dim(W), Q)$  is the set of injective maps, up to change of basis at gauged vertices, together with a compatible action by a nilpotent operator of  $\mathbb{C}^{w_n}$ :

$$\{ \mathbb{C}^{v_1} \hookrightarrow \mathbb{C}^{v_2} \hookrightarrow \dots \hookrightarrow \mathbb{C}^{v_n} \hookrightarrow \mathbb{C}^{w_n} \mid X(\mathbb{C}^{v_i}) \subseteq \mathbb{C}^{v_{i-1}} \text{ and } X^{n+1} = 0 \text{ for some } X \in \text{End}_{w_n}(\mathbb{C}). \}.$$

<sup>1</sup>Assume there is a linear injection  $J: \mathbb{C}^v \hookrightarrow \mathbb{C}^w$ , and suppose  $X \in \text{End}(\mathbb{C}^w)$  satisfies  $X(\mathbb{C}^w) \subseteq J(\mathbb{C}^v)$  and  $X^2 = 0$  (note that this implies  $X|_{J(\mathbb{C}^v)} = 0$ .) Setting  $I := J|_{J(\mathbb{C}^v)}^{-1} \circ X$ , we see that  $X = JI$ . Furthermore,  $IJ = J|_{J(\mathbb{C}^v)}^{-1} \circ X \circ J = J|_{J(\mathbb{C}^v)}^{-1} \circ X|_{J(\mathbb{C}^v)} = 0$ . So, any such  $X$  must have the desired form  $JI$ .

This is the Springer realization for the cotangent bundle of the partial flag variety  $T^*(\text{Fl}(v_1, v_2, \dots, v_n; \mathbb{C}^{w_n}))$ .<sup>2</sup>

**Remark 2.10.** In general, a Nakajima quiver variety for a type  $A_n$  quiver is a **type A Nakajima quiver variety**, and not all of these are isomorphic to cotangent bundles of partial flag varieties.

#### REFERENCES

- [Kir16] Alexander Kirillov, Jr. *Quiver representations and quiver varieties*, volume 174 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2016. doi:10.1090/gsm/174.

---

<sup>2</sup>Assume there is a linear injection  $J_n: \mathbb{C}^{v_n} \hookrightarrow \mathbb{C}^{w_n}$  and linear injections  $x_i: \mathbb{C}^{v_i} \hookrightarrow \mathbb{C}^{v_{i+1}}$  for  $i = 1, \dots, n$  with  $x_n := J_n$  and  $v_{n+1} := w_n$ . We will take  $\mathbb{C}^{v_0} := 0$ . Set  $x_{\{j,j-1,\dots,i\}} := x_j \circ x_{j-1} \circ \dots \circ x_i$  for  $j \geq i$ , and suppose  $X \in \text{End}(\mathbb{C}^{w_n})$  satisfies  $X(x_{\{n,n-1,\dots,i\}}(\mathbb{C}^{v_i})) \subseteq x_{\{n,n-1,\dots,i-1\}}(\mathbb{C}^{v_{i-1}})$  and  $X(\mathbb{C}^{w_n}) \subseteq J_n(\mathbb{C}^{v_n})$  and  $X^{n+1} = 0$ . Setting  $I_n := x_n|_{x_n(\mathbb{C}^{v_n})}^{-1} \circ X$ , we see that  $X = x_n I_n$ . Furthermore,  $I_n x_{\{n,n-1,\dots,i\}}(\mathbb{C}^{v_i}) = x_n|_{x_n(\mathbb{C}^{v_n})}^{-1} \circ X \circ x_{\{n,n-1,\dots,i\}}(\mathbb{C}^{v_i}) = x_n|_{x_n(\mathbb{C}^{v_n})}^{-1} \circ X|_{x_{\{n,n-1,\dots,i\}}(\mathbb{C}^{v_i})}(x_{\{n,n-1,\dots,i\}}(\mathbb{C}^{v_i})) \subseteq x_{\{n-1,\dots,i-1\}}(\mathbb{C}^{v_{i-1}})$ . So, any such  $X$  must have the desired form  $J_n I_n$ .