Generalized cohomology rings of rank 2 root systems

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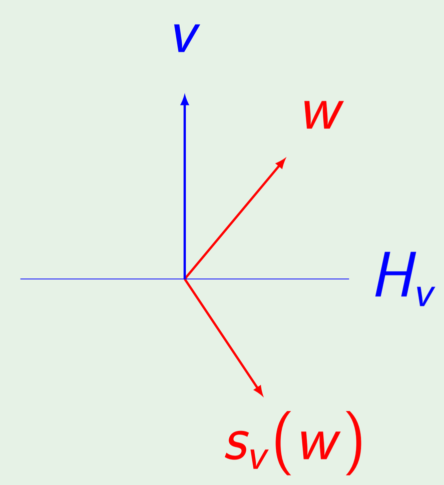
Reflections

Let $v \in \mathbb{R}^n$, and fix the standard inner product $(,)$ on $\mathbb{R}^n$. The **reflection** across the hyperplane H_v of v is the linear operator s_v on $\mathbb{R}^n$, defined by

$$s_v(w) = w - 2\frac{(v,w)}{(v,v)}v, \quad w \in \mathbb{R}^n.$$

Example



Root systems

A **root system** in $\mathbb{R}^n$ is a subset Σ of vectors called **roots**, satisfying:

- Σ is finite, nonempty, and does not contain 0.
- If $\alpha \in \Sigma$, then the only multiples of α in Σ are $\pm\alpha$.
- $s_\alpha(\Sigma) = \Sigma$ for every $\alpha \in \Sigma$.
- If $\alpha, \beta \in \Sigma$, then $s_\alpha(\beta) - \beta = n\alpha$ for some $n \in \mathbb{Z}$.

We call n the **rank** of the root system. There is always a basis for $\mathbb{R}^n$ in Σ, such that all roots are linear combinations of basis elements with coefficients all positive or all negative. Such a basis is a **simple system** $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ for Σ. The **Weyl group** W of Σ is the group generated by the reflections s_α, where $\alpha \in \Sigma$. Each root system has a **Dynkin type**, one of A, B, C, D, E, F, G.

Example

We illustrate the four root systems of rank 2, together with choice of simple system.

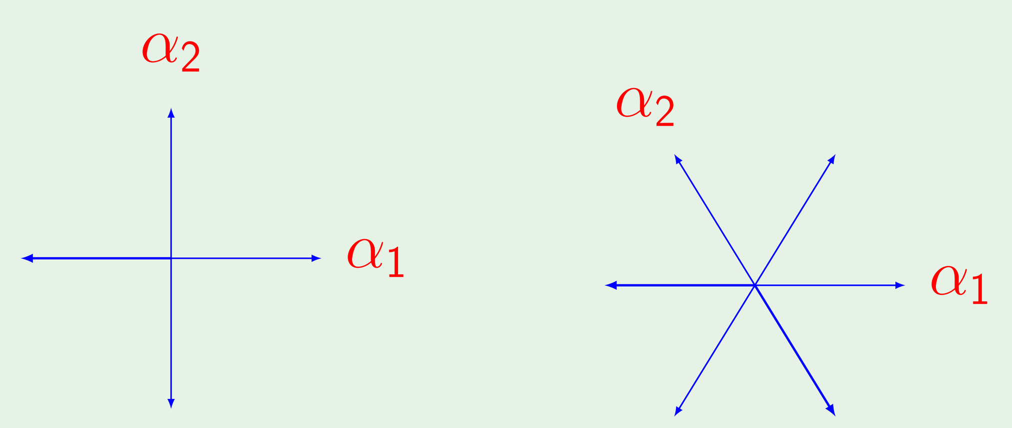


Figure: $A_1 \times A_1$ on the left and A_2 on the right.

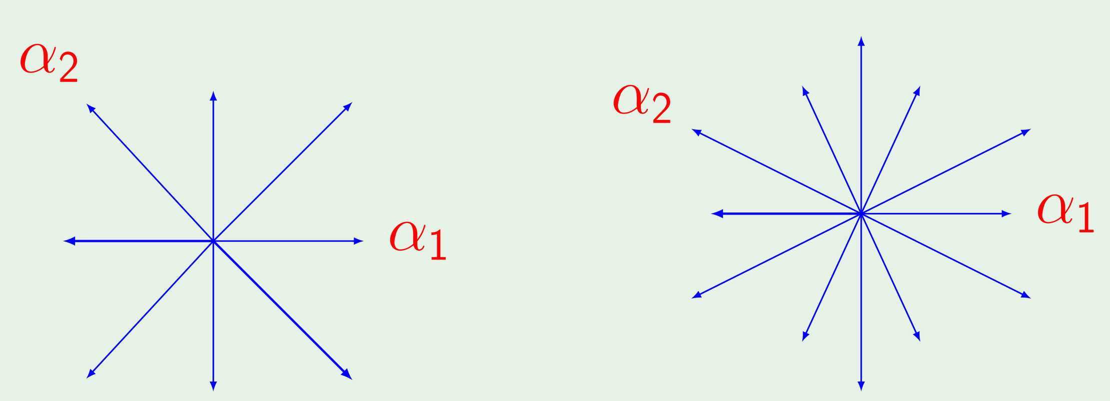


Figure: B_2 on the left and G_2 on the right.

Linear algebraic groups

A closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ is called a **linear algebraic group**. Every root system gives rise to a family of (semisimple, connected) linear algebraic groups. (In general, a semisimple, connected linear algebraic group corresponds to a **root datum**, which will not be discussed here.) Two notable types of groups in this family are the simply-connected and adjoint groups.

Example

Below we list the simply-connected and adjoint groups for classical types:

Root system	G^{sc}	G^{ad}
A_n	$\mathrm{SL}(n+1, \mathbb{C})$	$\mathrm{PGL}(n+1, \mathbb{C})$
B_n	$\mathrm{Spin}(2n+1, \mathbb{C})$	$\mathrm{SO}(2n+1, \mathbb{C})$
C_n	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{PSp}(2n, \mathbb{C})$
D_n	$\mathrm{Spin}(2n, \mathbb{C})$	$\mathrm{PGO}(2n, \mathbb{C})$

Formal group law

A **formal group law** over a commutative unital ring R is a formal power series $F(u,v) = \sum_{i,j\geq 0} a_{i,j} u^i v^j \in R[\![u,v]\!]$, satisfying:

1. $F(0,v) = v$; **2.** $F(u,v) = F(v,u)$; **3.** $F(F(u,v),w) = F(u,F(v,w))$.

Example

- The additive formal group law is $F(u,v) = u + v$.
- The multiplicative (periodic) formal group law is $F(u,v) = u + v - \beta uv$, where β is invertible in R.
- The universal formal group law is $F(u,v) = \sum_{i,j} a_{i,j} u^i v^j$, defined over the Lazard ring $\mathbb{L}$, which is the quotient of the free ring generated by the $a_{i,j}$ subject only to the relations imposed by the axioms of the formal group law.

Oriented cohomology rings

A **oriented cohomology theory** h^* is a 'functorial' map

$$\mathrm{h}^*\colon \{\text{smooth complex algebraic varieties}\} \to \{\text{graded commutative rings}\},$$

that satisfies various 'cohomological' axioms.

To every oriented cohomology theory, one can associate a formal group law.

Example

Below are the formal group law associated with some cohomology theories.

Oriented cohomology theory	Formal group law
Chow rings CH^*	Additive over $R = \mathbb{Z}$
K-theory K^0	Multiplicative over $R = \mathbb{Z}[\beta, \beta^{-1}]$
Cobordism Ω^*	Universal over Lazard ring

New results

Let h^* be an oriented cohomology theory, whose formal group law is $F(u,v) = \sum_{i,j} a_{i,j} u^i v^j$ over the ring $R = \mathrm{h}^*(\mathrm{pt})$ (where pt is a point). Minimal presentations for the cobordism rings for the simple rank 1 and 2 groups are listed in the table below:

Rank	Σ	Λ	G	$\mathrm{h}^*(G)$	$K^0(G)$	$\mathrm{CH}^*(G)$
1	A_1	$\Lambda^{\mathrm{ad}}_{A_1}$	$\mathrm{PGL}(2,k)$	$\frac{R[x]}{(2x,x^2)}$	$\frac{R[x]}{(2x,x^2)}$	$\frac{R[x]}{(2x,x^2)}$
		$\Lambda^{\mathrm{sc}}_{A_1}$	$\mathrm{SL}(2,k)$	R	R	R
2	A_2	$\Lambda^{\mathrm{ad}}_{A_2}$	$\mathrm{PGL}(3,k)$	$\frac{R[x]}{(3x,x^3)}$	$\frac{R[x]}{(3x,x^3)}$	$\frac{R[x]}{(3x,x^3)}$
		$\Lambda^{\mathrm{sc}}_{B_2}$	$\mathrm{SL}(3,k)$	R	R	R
2	B_2	$\Lambda^{\mathrm{ad}}_{B_2}$	$\mathrm{SO}(5,k)$	$\frac{R[x]}{(2x-a_{11}x^2,2x^2,x^4)}$	$\frac{R[x]}{(2x-x^2,x^3)}$	$\frac{R[x]}{(2x,x^4)}$
		$\Lambda^{\mathrm{sc}}_{B_2}$	$\mathrm{Spin}(5,k)$	R	R	R
2	G_2	$\Lambda^{\mathrm{ad}}_{G_2} = \Lambda^{\mathrm{sc}}_{G_2}$	G_2	$\frac{R[x]}{(a_{11}x,2x,x^2)}$	R	$\frac{R[x]}{(2x,x^2)}$

References

1. Gandhi, Raj. *Oriented cohomology rings of the semisimple linear algebraic groups of ranks* 1 *and* 2. M.Sc. Thesis. University of Ottawa library, 2021.