Algebraic Cobordism Rings of Wonderful Varieties and Matroids

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Matroids and wonderful varieties

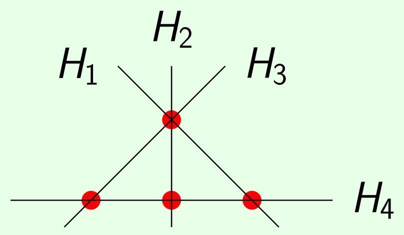
A **matroid** is a pair $M = (\mathcal{I}, E)$, where E is a finite set and $\mathcal{I}$ is a set of subsets of E s.t.:

1. The empty set $\emptyset \in \mathcal{I}$.
2. If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.
3. If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then $I_1 \cup \{e\} \in \mathcal{I}$ for some $e \in I_2 \setminus I_1$.

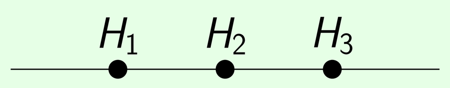
The elements of $\mathcal{I}$ are **independent sets**. The **rank** $r(S)$ of $S \subseteq E$ is the size of the largest independent set in S. A subset F of E is a **flat** if $r(F \cup i) > r(F)$ for all $i \in E \setminus F$.

Let $\mathcal{H} = \{H_i\}_{i=1}^k$ be an arrangement of linear hyperplanes in $L := \mathbb{C}^n$, s.t. $\cap_{i=1}^k H_i = (0)$. Let n_{H_i} be the vector normal to H_i. The matroid $M_{\mathcal{H}}$ of $\mathcal{H}$ has ground set $E = \{H_i\}_{i=1}^k$, and a subset $S \subseteq E$ is **independent** if $\{n_i\}_{i\in S}$ is linearly independent. The **rank** of $S \subseteq E$ is $r(S) := \dim(\operatorname{span}\{n_i\}_{i\in S})$, and S is a **flat** if $r(S \cup H_i) > r(S)$ for every $H_i \in E \setminus S$.

Let $\mathbb{P}\mathcal{H} = \{\mathbb{P}(H_i)\}_{i\in E}$ be the projectivization of $\mathcal{H}$ inside $\mathbb{P}(L)$. There is a smooth simple normal crossings compactification $W_{\mathcal{H}}$ of the open set $\mathbb{P}(L) \setminus \cup_{i\in E}\mathbb{P}(H_i)$ called the **wonderful variety** of $\mathcal{H}$, which is an iterated blow-up of $\mathbb{P}(L)$ along the flats of $M_{\mathcal{H}}$. It was constructed by de Concini-Procesi [DP]. The exceptional divisors D_G of the blow-up are indexed by the proper, non-empty flats G of $M_{\mathcal{H}}$.



Flats: $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,4\}$, $\{2,4\}$, $\{3,4\}$, $\{1,2,3\}$, $\{1,2,3,4\}$.



Flats: $\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}$.

Formal group laws

Let R be a commutative ring. A **formal group law** (R, F) is a formal power series $F(x,y) = \sum_{i,j\geq 0} a_{i,j}x^iy^j \in R[\![x,y]\!]$, such that for all $w, v, z \in R[\![x,y]\!]$:

1. $F(0,v) = v$,
2. $F(w,v) = F(v,w)$, and
3. $F(w, F(v,z)) = F(F(w,v),z)$.

Given a formal group law (R,F), there is a unique formal power series

$$-_F x := -x - c_2x^2 - c_3x^3 - \cdots \in R[\![x]\!]$$

called the **formal inverse** of (R,F) that satisfies $F(x, -_F x) = 0$.

For a formal group law (R,F), elements $g_1, \ldots, g_k \in R[\![x_1,\ldots,x_n]\!]$, and $a \in \mathbb{Z}$, we define

$$g_1 +_F g_2 := F(g_1,g_2); \quad a \cdot_F g_1 := \underbrace{g_1 +_F g_1 +_F \cdots +_F g_1}_{a \text{ times}}; \quad \sum_{i\in[k]}^{F} g_i := g_1 +_F g_2 +_F \cdots +_F g_k.$$

Examples:

- The **additive formal group law** over $\mathbb{Z}$ is $F_A(x,y) = x + y$, and $-_{F_A}x = -x$.
- The **multiplicative-periodic formal group law** over $\mathbb{Z}[\beta,\beta^{-1}]$ is $F_M(x,y) = x + y - \beta xy$, and $-_{F_M}x = \frac{-x}{1-\beta x}$.
- The **universal formal group law** is defined by $F_U(x,y) = \sum_{i,j\geq 0} a_{i,j}x^iy^j \in \mathbb{L}^*[\![x,y]\!]$, where $\mathbb{L}^*$ is the **Lazard ring**, i.e., the the quotient of the free $\mathbb{Z}$-algebra with generators $a_{i,j}$, $i,j \geq 0$, mod the relations imposed by the axioms of a formal group law.

Combinatorics of Chern classes

An **oriented cohomology theory** (OCT) is a contravariant functor

$$\mathrm{h}^*\colon \{\text{cat. of smooth varieties}/\mathbb{C}\} \to \{\text{cat. of commutative, graded rings}\}$$

satisfying cohomological axioms. They were introduced by Levine and Morel [LM].

Important facts:

- For a rank r vector bundle E on a smooth variety X, there are Chern classes
$$c_i^{\mathrm{h}^*}(E) \in \mathrm{h}^i(X), \quad i = 0, \ldots, r.$$
- Every OCT h^* comes with a formal group law F_{h^*} over $R_{\mathrm{h}^*} := \mathrm{h}^*(\mathrm{point})$, characterized by the **Quillen formula**: for any two line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ on a smooth variety X,
$$c_1^{\mathrm{h}^*}(\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2) = F_{\mathrm{h}^*}(c_1^{\mathrm{h}^*}(\mathcal{L}_1), c_1^{\mathrm{h}^*}(\mathcal{L}_2)).$$

Examples:

- The **Chow theory** $\mathrm{CH}^*(-)$ which sends a smooth variety X to its Chow ring $\mathrm{CH}^*(X)$ is an OCT. If $\mathcal{L} = \mathcal{O}_X(D)$ is the line bundle on X corresponding to a smooth divisor D, then $c_1(\mathcal{L}) = [D] \in \mathrm{CH}^*(X)$. The formal group law of CH^* is $(\mathbb{Z}, F_A)$:
$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2).$$
- Let $K^0(X)$ be the Grothendieck ring of algebraic vector bundles on a smooth variety X. The **graded** K**-theory** $K^*(-)$ which sends a smooth variety X to the ring $K^0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta,\beta^{-1}]$, with $\deg(\beta) = -1$, is an OCT. If $\mathcal{L}$ is a line bundle on X, then $c_1(\mathcal{L}) = \beta^{-1}(1 - [\mathcal{L}^\vee]) \in K^*(X)$. The formal group law of $K^*(-)$ is $(\mathbb{Z}[\beta,\beta^{-1}], F_M)$:
$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) - \beta \cdot c_1(\mathcal{L}_1)\cdot c_1(\mathcal{L}_2).$$

Algebraic cobordism

The **algebraic cobordism** functor $\Omega^*(-)$ is an OCT that sends a smooth variety X to the algebraic cobordism ring $\Omega^*(X)$ generated by classes of projective morphisms $Y \to X$, where Y is smooth, quasi-projective, and irreducible, modulo relations. Given a smooth divisor D on X, we have $c_1(\mathcal{O}(D)) = [D \hookrightarrow X]$, and the formal group law of $\Omega^*(-)$ is $(\mathbb{L}^*, F_U)$.

Given a smooth variety X, there are natural ring homomorphisms

$$\Omega^*(X) \to \mathrm{CH}^*(X), \quad \text{and} \quad \Omega^*(X) \to K^*(X),$$

which induce ring isomorphisms

$$\Omega^*(X) \otimes_{\mathbb{L}^*} \mathbb{Z} \simeq \mathrm{CH}^*(X) \quad \text{and} \quad \Omega^*(X) \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta,\beta^{-1}] \simeq K^*(X). \tag{1}$$

Example:

The uniform matroid $U_{2,3}$ is the matroid on ground set $E = \{1,2,3\}$, with flats

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}.$$

Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$. The Bergman fan of $U_{2,3}$ lies in $\mathbb{R}^3/(e_1+e_2+e_3)$:

$$\text{Rays:} \quad \sigma_1 = \operatorname{span}(e_1), \quad \sigma_2 = \operatorname{span}(e_2), \quad \sigma_3 = \operatorname{span}(e_3) = \operatorname{span}(-e_1-e_2).$$

The toric variety of this fan is $X_{U_{2,3}} = \mathbb{P}^2 \setminus \{3 \text{ distinct points}\}$, and the divisors of $X_{U_{2,3}}$ corresponding to the rays are D_{σ_1}, D_{σ_2}, D_{σ_3}. Set $x_i := [D_{\sigma_i} \hookrightarrow X_{U_{2,3}}] \in \Omega^*(X_{U_{2,3}})$. Then:

$$\Omega^*(X_{U_{2,3}}) \simeq \frac{\mathbb{L}^*[x_1,x_2,x_3,x_4]}{(x_1\cdot x_2,\ x_1\cdot x_3,\ x_2\cdot x_3,\ x_4 +_{F_U}(-_{F_U}x_1),\ x_4 +_{F_U}(-_{F_U}x_2),\ x_4 +_{F_U}(-_{F_U}x_3))}.$$

One can show that $\Omega^*(X_{U_{2,3}}) \simeq \mathbb{L}^*[x]/(x^2)$. By (1), we can replace the pair $(\mathbb{L}^*, F_U)$ with

- $(\mathbb{Z}, F_A)$ everywhere to get the Feichtner-Yuzvinsky presentation for $\mathrm{CH}^*(X_{U_{2,3}})$ ([FY]).
- $(\mathbb{Z}[\beta,\beta^{-1}], F_M)$ everywhere and set $\beta = 1$ to obtain the Feichtner-Yuzvinsky presentation for $K^0(X_{U_{2,3}})$ ([LLPP]).

Algebraic cobordism rings of wonderful varieties and matroids

We will now give two descriptions for the algebraic cobordism ring of $W_{\mathcal{H}}$, where $\mathcal{H} = \{H_i\}_{i\in E}$ is a hyperplane arrangement with matroid $M_{\mathcal{H}}$. Consider the polynomial rings $\mathcal{T}_{M_{\mathcal{H}}} := \mathbb{L}^*[x_G : \ G \text{ a non-empty flat of } M_{\mathcal{H}}]$, $\quad \mathcal{S}_{M_{\mathcal{H}}} := \mathbb{L}^*[h_G : \ G \text{ a non-empty flat of } M_{\mathcal{H}}]$.

Feichtner-Yuzvinsky presentation:

- Define the following two ideals in $\mathcal{T}_{M_{\mathcal{H}}}$:
$$\mathcal{I}_1^{M_{\mathcal{H}}} := \langle x_Gx_{G'} : \ G \text{ and } G' \text{ are incomparable}\rangle \quad \text{and} \quad \mathcal{I}_2^{M_{\mathcal{H}}} := \left\langle \sum_{j_0\in G}^{F_U} x_G : j_0 \in E \right\rangle.$$
- Fix any $i_0 \in E$. There is an $\mathbb{L}^*$-algebra isomorphism
$$\frac{\mathcal{T}_{M_{\mathcal{H}}}}{\mathcal{I}_1^{M_{\mathcal{H}}} + \mathcal{I}_2^{M_{\mathcal{H}}}} \to \Omega^*(W_{\mathcal{H}}), \quad x_G \mapsto \begin{cases} [D_G \hookrightarrow W_{\mathcal{H}}], & G \neq E; \\ \sum_{i_0\in H}^{F_U}(-_{F_U}[D_H \hookrightarrow W_{\mathcal{H}}]), & G = E, \end{cases}$$
where in the case $G = E$, the formal sum is over *all* flats H that contain $i_0 \in E$.

Simplicial presentation:

- Define the following two ideals in $\mathcal{S}_{M_{\mathcal{H}}}$:
$$\mathcal{J}_1^{M_{\mathcal{H}}} := \langle (h_G - h_{G\vee H})(h_H - h_{G\vee H}) : G, H \text{ non-empty flats of } M_{\mathcal{H}}\rangle$$
$$\mathcal{J}_2^{M_{\mathcal{H}}} := \langle h_G : \operatorname{rank}_{M_{\mathcal{H}}}(G) = 1\rangle.$$
- There are $\mathbb{L}^*$-algebra isomorphisms
$$\frac{\mathcal{S}_{M_{\mathcal{H}}}}{\mathcal{J}_1^{M_{\mathcal{H}}} + \mathcal{J}_2^{M_{\mathcal{H}}}} \xrightarrow{\Phi} \frac{\mathcal{T}_{M_{\mathcal{H}}}}{\mathcal{I}_1^{M_{\mathcal{H}}} + \mathcal{I}_2^{M_{\mathcal{H}}}} \simeq \Omega^*(W_{\mathcal{H}})$$
where Φ is defined by declaring $\Phi(h_G) := \sum_{H\supseteq G}^{F_U}(-_{F_U}x_H)$.

Consequences:

- Via (1), the above presentations specialize to the Feichtner-Yuzvinsky and simplicial presentations for $\mathrm{CH}^*(W_{\mathcal{H}})$ and $K^0(W_{\mathcal{H}})$ of [FY], [LLPP], [BES].
- Φ is an $\mathbb{L}^*$-algebra isomorphism $\Phi\colon \mathbb{L}^* \otimes_{\mathbb{Z}} \mathrm{CH}^*(W_{\mathcal{H}}) \xrightarrow{\simeq} \Omega^*(W_{\mathcal{H}})$, which specializes to the exceptional isomorphism $K^0(W_{\mathcal{H}}) \xrightarrow{\simeq} \mathrm{CH}^*(W_{\mathcal{H}})$ of [BEST], [LLPP].
- The two presentations and map Φ are combinatorial and generalize to the algebraic cobordism ring of the toric variety of the Bergman fan of any loopless matroid.
- $\Omega^*(W_{\mathcal{H}})$ is isomorphic to the complex cobordism ring of $W_{\mathcal{H}}$, and two wonderful varieties $W_{\mathcal{H}_1}$ and $W_{\mathcal{H}_2}$ such that $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same matroid define the same class in the complex (and algebraic) cobordism ring of the point.

References

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