Connective motivic Chern classes of Schubert cells

Raj Gandhi, Cornell University

The connective motivic Segre classes

Consider the following $\mathbb{Z}[\beta][q^{\pm 1}]$-linear operators on $\mathbb{Z}[\beta][e^{\pm y_1}, \dots, e^{\pm y_n}, q^{\pm 1}]$:

$$\partial_i := \frac{\beta(1-q^2)}{1-e^{y_{i+1}-y_i}} + \frac{\beta(1-q^2)+q^2(1-e^{y_i-y_{i+1}})}{1-e^{y_i-y_{i+1}}} r_i,$$

which satisfy:

1. $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ for all $i, j = 1, \dots, n-1$ such that $|i-j| > 1$.
2. $\partial_i \circ \partial_{i+1} \circ \partial_i = \partial_{i+1} \circ \partial_i \circ \partial_{i+1}$ for all $i = 1, \dots, n-2$.
3. $(\partial_i + q^2) \circ (\partial_i + (q^2\beta - q^2 - \beta)) = 0$ for all $i = 1, \dots, n-1$.

Let Γ be the set of 01 sequences with k 1s and $n-k$ 0s. Consider the ring

$$R := \bigoplus_{\lambda \in \Gamma} \mathrm{Frac}(\mathbb{Z}[\beta][e^{\pm y_1}, \dots, e^{\pm y_n}, q^{\pm 1}]).$$

The group S_n acts on R by $w \cdot (f_\lambda)_\lambda := (w(f_\lambda))_{w(\lambda)}$, $w \in S_n$. Set $\omega := 11\cdots 100\cdots 0$.

$$S_\omega|_\lambda := \begin{cases} \prod_{i>j:\lambda_i<\lambda_j} \frac{1-e^{y_i-y_j}}{\beta(1-q^2)+q^2(1-e^{y_i-y_j})}, & \text{if } \lambda = \omega; \\ 0, & \text{otherwise.} \end{cases}$$

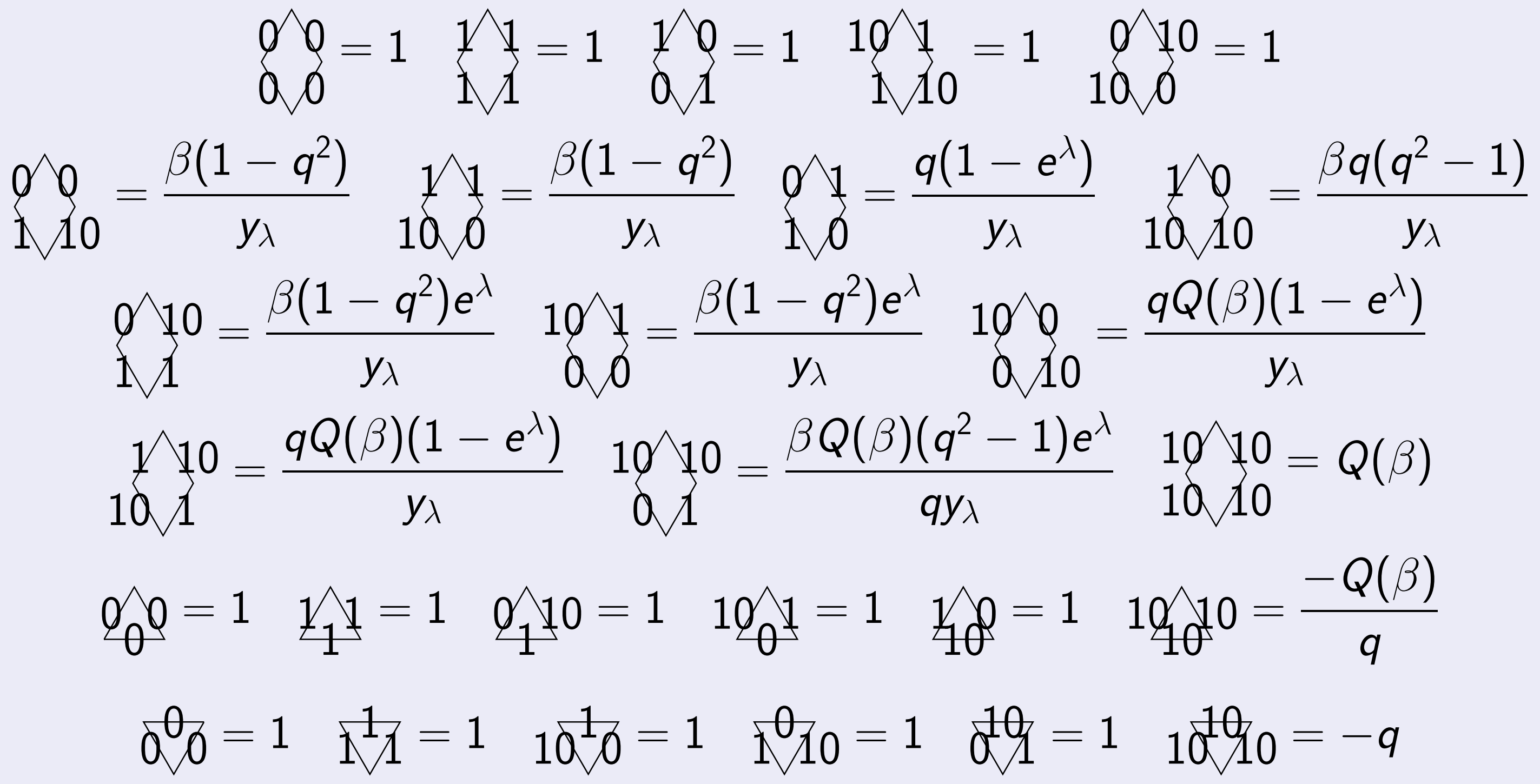
The other S_λ are defined by the rule $S_{w^{-1}(\omega)} := \partial_w(S_\omega)$. At $\beta = 1$, the S_λ are the motivic Segre classes of Schubert cells. At $\beta \to 0$, the S_λ are the SSM classes of Schubert cells.

Knutson-Tao puzzles

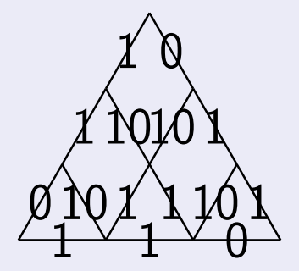
Define $Q(\beta) := q^2 + \beta - q^2\beta$. For $\lambda \in \mathbb{Z}[y_1, \dots, y_n]$, set

$$y_\lambda := \beta(1-q^2) + q^2(1-e^\lambda).$$

A **puzzle** with side labels λ, μ, ν in Γ is a triangle with side labels λ, μ, ν that is tiled by the following **puzzle pieces** with edge labels 0, 1, and 10. Each puzzle piece comes with a function $\mathbb{Z}[y_1, \dots, y_n] \to \mathrm{Frac}(\mathbb{Z}[\beta][e^{\pm y_1}, \dots, e^{\pm y_n}, q^{\pm 1}])$ called its **fugacity**:



The bottom row of a puzzle is always tiled by the triangle puzzle pieces, and the rest of the puzzle is tiled by (not rotated) rhombus puzzle pieces. The **fugacity of a puzzle** is the product of fugacities of the rhombi and triangles that tile it. A triangle tile has constant fugacity, whereas the fugacity of a rhombus tile depends on its position in the puzzle. A rhombus tile that lies in the i-th southwest-to-northeast diagonal and the j-th northwest-to-southeast diagonal depends on $\lambda = -(y_i - y_j)$. For example, the fugacity of the puzzle below is $\frac{qQ(\beta)(1-e^{y_2-y_1})}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})} \cdot \frac{\beta q(q^2-1)}{\beta(1-q^2)+q^2(1-e^{y_3-y_1})} \cdot 1$.



Multiplying the connective Segre classes

We will use the notation $\lambda \underset{\nu}{\triangle} \mu$ to mean the sum over the fugacities of all possible puzzles with the prescribed boundary labels. The **length** $l(\lambda)$ of an element $\lambda \in \Gamma$ is the minimal number of simple transpositions needed to turn λ into $0^{n-d}1^d$.

THEOREM

The product of two classes $q^{l(\lambda)}S_\lambda$ and $q^{l(\mu)}S_\mu$ is given by the "puzzle" formula

$$(q^{l(\lambda)}S_\lambda)(q^{l(\mu)}S_\mu) = \sum_\nu \lambda \underset{\nu}{\triangle} \mu \; (q^{l(\nu)}S_\nu).$$

The structure coefficients **positive**, and are sums of products of these factors:

$$-q^{\pm} \qquad Q(\beta) \qquad e^{y_j-y_i} \qquad \frac{\beta(1-q^2)}{\beta(1-q^2)+q^2(1-e^{y_j-y_i})} \qquad -\frac{1-e^{y_j-y_i}}{\beta(1-q^2)+q^2(1-e^{y_j-y_i})}$$

Example

Let us compute the classes $q \cdot S_{10}$ and S_{01} for $T^*(\mathrm{Gr}(1,2))$. The classes are

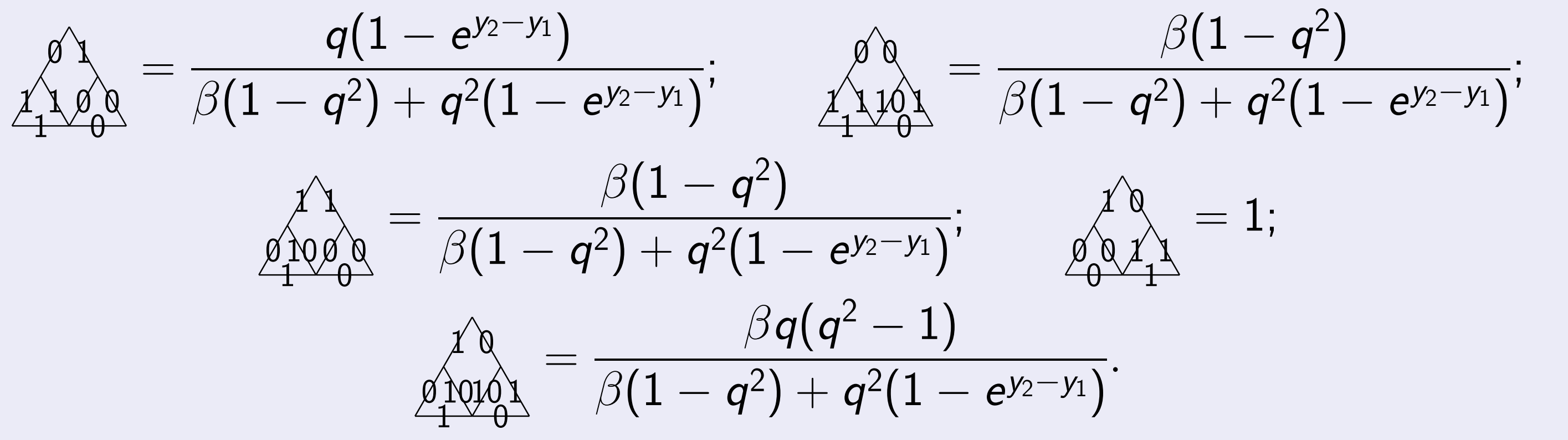
$$q \cdot S_{10} = q \cdot \left[0, \frac{1-e^{y_2-y_1}}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}\right]; \qquad S_{01} = (\tfrac{1}{q}\partial_\alpha)(q \cdot S_{10}) = \left[1, \frac{\beta(1-q^2)}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}\right].$$

Let us now multiply classes together:

$$(q \cdot S_{10})^2 = \frac{q(1-e^{y_2-y_1})}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}(q \cdot S_{10}); \quad (q \cdot S_{10})S_{01} = S_{01}(q \cdot S_{10}) = \frac{\beta(1-q^2)}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}(q \cdot S_{10});$$

$$S_{01}^2 = S_{01} + \frac{\beta q(q^2-1)}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}(q \cdot S_{10}).$$

The puzzles give the same result:



R-matrix recursion

The classes $\overline{S}_\lambda := q^{\ell(\lambda)}S_\lambda$ are characterized by the following properties:

1. *Triangularity:* $\overline{S}_\lambda|_\sigma = 0$ unless $\sigma \geq \lambda$ in the Bruhat order.
2. *Diagonal entries:*
$$\overline{S}_\lambda|_\lambda = \prod_{i<j:\lambda_i>\lambda_j} \frac{q(1-e^{y_j-y_i})}{Q(\beta,q)-q^2e^{y_j-y_i}}$$
3. *Exchange relation:*
$$r_i\overline{S}_\lambda|_{\sigma r_i} = \begin{cases} \frac{\beta(1-q^2)}{Q(\beta,q)-q^2e^{y_i-y_{i+1}}}\overline{S}_\lambda|_\sigma + \frac{qQ(\beta,q)(1-e^{y_i-y_{i+1}})}{Q(\beta,q)-q^2e^{y_i-y_{i+1}}}\overline{S}_{\lambda r_i}|_\sigma & \lambda_i < \lambda_{i+1} \\ \overline{S}_\lambda|_\sigma & \lambda_i = \lambda_{i+1} \\ \frac{\beta(1-q^2)(e^{y_i-y_{i+1}})}{Q(\beta,q)-q^2e^{y_i-y_{i+1}}}\overline{S}_\lambda|_\sigma + \frac{q(1-e^{y_i-y_{i+1}})}{Q(\beta,q)-q^2e^{y_i-y_{i+1}}}\overline{S}_{\lambda r_i}|_\sigma & \lambda_i > \lambda_{i+1} \end{cases}$$

This recursion is derived from the representation theory of the **multiparameter quantum group** of type $\widehat{\mathfrak{a}}_2$.

Multi-parameter quantum group

The **multi-parameter quantum group** $U_{\mathbf{q}}(\widehat{a}_n)$ is the associative unital algebra over $\mathbb{K} = \mathbb{Q}(q_{i,j})$ generated by elements E_i, F_i, $K_i^{(1)}$, $(K_i^{(1)})^{-1}$, $K_i^{(2)}$, $(K_i^{(2)})^{-1}$, where $i = 0, 1, \dots, n$, subject to the relations:

- $K_i^{(2)}(K_i^{(2)})^{-1} = (K_i^{(2)})^{-1}K_i^{(2)} = 1, \qquad K_i^{(1)}(K_i^{(1)})^{-1} = (K_i^{(1)})^{-1}K_i^{(1)} = 1,$
- $K_i^{(1)}K_j^{(2)} = K_j^{(2)}K_i^{(1)}, \qquad K_i^{(1)}K_j^{(1)} = K_j^{(1)}K_i^{(1)}, \qquad K_i^{(2)}K_j^{(2)} = K_j^{(2)}K_i^{(2)},$
- $K_i^{(1)}E_j(K_i^{(1)})^{-1} = q_{i,j}E_j, \qquad (K_i^{(2)})^{-1}E_jK_i^{(2)} = q_{j,i}^{-1}E_j,$
- $K_i^{(1)}F_j(K_i^{(1)})^{-1} = q_{i,j}^{-1}F_j, \qquad (K_i^{(2)})^{-1}F_jK_i^{(2)} = q_{j,i}F_j,$
- $[E_i, F_j] = \delta_{i,j}\frac{q_{i,i}}{q_{i,i}-1}\left(K_i^{(1)} - (K_i^{(2)})^{-1}\right),$
- $E_i^2E_j - q_{i,j}(1+q_{i,i})E_iE_jE_i + \frac{q_{i,j}}{q_{j,i}}E_jE_i^2 = 0, \qquad i \neq j,$
- $\frac{q_{i,j}}{q_{j,i}}F_i^2F_j - q_{i,j}(1+q_{i,i})F_iF_jF_i + F_jF_i^2 = 0, \qquad i \neq j.$

THEOREM

The matrix of puzzle fugacities is an R-matrix for $U_{\mathbf{q}}(\widehat{\mathfrak{a}}_2)$: $R(z) = \left(\mathrm{fug}\left({}^{k}_{i}\lozenge^{\ell}_{j}\right)\right)$

Equivariant connective K-theory

Equivariant connective K-theory $\mathrm{CK}(-)$ is a generalized cohomology theory that interpolates between equivariant Chow rings ($\beta = 0$) and equivariant K-theory ($\beta = 1$). Let $T = (\mathbb{C}^\times)^n$ be a torus, and X a smooth T-variety.

WORK IN PROGRESS (with Anubhav Nanavaty)

- There is a **connective motivic Chern transformation**
$$\mathrm{CMC}\colon K_0^T(\mathrm{Var}/X) \to \mathrm{CK}_T(X) \otimes \mathbb{Z}[y].$$
- Consider the Grassmannian $\mathrm{Gr}(k,n) = \sqcup_{\lambda \in \Gamma} X_\lambda^\circ$, where X_λ° are the Schubert cells. The torus T acts on each Schubert cell X_λ°. We have
$$\mathrm{CMC}(X_\lambda^\circ) = S_\lambda.$$
- There is an Atiyah-Segal isomorphism between $\mathrm{CK}_T(X)$ and a quotient of the T-equivariant algebraic cobordism ring of X, when X is equivariantly filtrable.

FUTURE WORK

- Compute the product of two S_λ classes using **quantum multiplication** in $\mathrm{CK}_T(X_\lambda^\circ)$.
- Study torus-equivariant connective K-rings of Nakajima quiver varieties. Are these rings representations for multiparameter quantum groups?
- More generally, study torus-equivariant algebraic cobordism rings of Nakajima quiver varieties. What Hopf algebra will arise?
- Can stable envelopes be defined for the equivariant connective K-rings of symplectic resolutions?
- Generalize the puzzle formula to 2, 3, and 4-step flag varieties, and separated descents.