

Algebraic cobordism rings of wonderful varieties and matroids

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What is this talk about?

Wonderful variety $W_{\mathcal{H}}$ of a hyperplane arrangement $\mathcal{H}$ is well-studied.

Remarkably, the Chow and K -rings of $W_{\mathcal{H}}$ are **integrally isomorphic**.

Algebraic cobordism Ω^* is the **universal** oriented cohomology theory.

The algebraic cobordism ring of $W_{\mathcal{H}}$ **specializes** to the Chow and K -ring.

Ethan and I study $\Omega^*(W_{\mathcal{H}})$ and find an $\Omega^*(\text{point})$ -algebra **isomorphism**:

$$\Omega^*(W_{\mathcal{H}}) \simeq \text{CH}^*(W_{\mathcal{H}}) \otimes_{\mathbb{Z}} \Omega^*(\text{point}),$$

which **specializes** to the ring isomorphism $K(W_{\mathcal{H}}) \simeq \text{CH}^*(W_{\mathcal{H}})$.

Part 1:

Formal group laws and
algebraic cobordism

Let's start with addition

Addition $F_A(x, y) := x + y$ has nice properties:

- ① Unit: $F_A(0, y) = y$ and $F_A(x, 0) = x$.
- ② Commutative: $F_A(x, y) = F_A(y, x)$.
- ③ Associative: $F_A(x, F_A(y, z)) = F_A(F_A(x, y), z)$.

Let's "axiomatize" power series with these properties:

$F(x, y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j \in R[[x, y]]$ is a **formal group law** if

- ① $F(x, 0) = x$ and $F(0, y) = y$ [$\implies F(x, y) = x + y + \sum_{i,j \geq 1} a_{i,j} x^i y^j$]
- ② $F(x, y) = F(y, x)$ [$\implies a_{i,j} = a_{j,i}$]
- ③ $F(x, F(y, z)) = F(F(x, y), z)$ [$\implies$ complicated relations among $a_{i,j}$]

Let us use the notation $x +_F y := F(x, y)$.

Practice with the group laws

The **multiplicative group law** satisfies $x +_{F_M} y := x + y - xy$.

There are very natural equations in $\mathbb{Z}[t_1, t_2, t_3, \dots, t_n]$:

$$\begin{aligned} 1 - (1 - t_1)(1 - t_2) &= t_1 + t_2 - t_1 t_2 \\ &= t_1 +_{F_M} t_2. \end{aligned}$$

$$\begin{aligned} 1 - (1 - t_1)(1 - t_2)(1 - t_3) &= t_1 + t_2 + t_3 - t_1 t_2 - t_1 t_3 - t_2 t_3 + t_1 t_2 t_3 \\ &= (t_1 + t_2 - t_1 t_2) + t_3 - (t_1 + t_2 - t_1 t_2) t_3 \\ &= (t_1 +_{F_M} t_2) + t_3 - (t_1 +_{F_M} t_2) t_3 \\ &= t_1 +_{F_M} t_2 +_{F_M} t_3 \end{aligned}$$

By induction:

$$1 - \prod_{i=1}^n (1 - t_i) = t_1 +_{F_M} t_2 +_{F_M} \cdots +_{F_M} t_n.$$

Inverses

Every group law F comes with an **inverse** $-_F$:

$$x +_F (-_F x) = 0.$$

Additive inverse law:

$$\begin{aligned} 0 &= x +_{F_A} (-_{F_A} x) \\ &= x + (-_{F_A} x). \end{aligned}$$

Thus, $(-_{F_A} x) = -x$.

Multiplicative inverse law:

$$\begin{aligned} 0 &= x +_{F_M} (-_{F_M} x) \\ &= x + (-_{F_M} x) - x(-_{F_M} x) \\ &= x + (-_{F_M} x)(1 - x). \end{aligned}$$

Thus, $(-_{F_M} x) = \frac{-x}{1-x} = -x(1 + x + x^2 + \dots)$.

The "universal" addition

The **Lazard ring** is

$$\mathbb{L}^* := \mathbb{Z}[a_{i,j}]/(\text{FGL relations}), \quad \deg(a_{i,j}) = 2(i+j-1).$$

Theorem (Lazard 1955)

The ring $\mathbb{L}^*$ is isomorphic to a graded polynomial ring

$$\mathbb{Z}[x_1, x_2, x_3, \dots] \quad \text{with } \deg(x_i) = 2i.$$

The **universal formal group law** is

$$F_U(x, y) = x + y + \sum_{i,j \geq 1} a_{i,j} x^i y^j \in \mathbb{L}^*[[x, y]].$$

Any FGL $F(x, y)$ can be recovered from $F_U(x, y)$ by specializing the $a_{i,j}$:

$$a_{i,j} = 0 \text{ for all } i, j \geq 1 \implies \text{additive } F_A(x, y) = x + y.$$

$$a_{1,1} = -1 \text{ and } a_{i,j} = 0 \text{ otherwise} \implies \text{multiplicative } F_M(x, y) = x + y - xy.$$

Why should we care about FGLs?

A cohomology theory with Chern classes is a functor satisfying natural axioms:

$$\{\textit{top. spaces/smooth schemes}\} \rightarrow \{\textit{commutative (graded) rings}\}$$

Cohomology theory $\rightsquigarrow$ formal group law.

Singular cohomology/Chow rings $\rightsquigarrow$ **additive FGL** $F_A(x, y) = x + y$.

K-theory $\rightsquigarrow$ **multiplicative FGL** $F_M(x, y) = x + y - xy$.

Elliptic cohomology $\rightsquigarrow$ **elliptic FGL**.

Algebraic/complex cobordism $\rightsquigarrow$ **universal FGL**.

How do FGLs arise from geometry?

Let X be a smooth complex variety, with the group of divisors $\mathrm{Div}(X)$ and Picard group $\mathrm{Pic}(X)$. There is a **group isomorphism**:

$$c_1^{\mathrm{CH}^*} : \mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Div}(X) / \sim, \quad \mathcal{L} \simeq \mathcal{O}_X(D) \mapsto c_1^{\mathrm{CH}^*}(\mathcal{O}_X(D)) = [D].$$

Chow rings: $\mathrm{CH}^*(X)$ is generated by classes of subvarieties of X , and

$$\begin{aligned} c_1^{\mathrm{CH}^*}(\mathcal{O}_X(D_1) \otimes_X \mathcal{O}_X(D_2)) &= c_1^{\mathrm{CH}^*}(\mathcal{O}_X(D_1)) + c_1^{\mathrm{CH}^*}(\mathcal{O}_X(D_2)) \\ &= F_A(c_1^{\mathrm{CH}^*}(\mathcal{O}_X(D_1)), c_1^{\mathrm{CH}^*}(\mathcal{O}_X(D_2))). \end{aligned}$$

K-theory Chern class arithmetic

K-theory: $K(X)$ is generated by isomorphism classes of algebraic vector bundles $[\mathcal{V}]$ on X , modulo the relation

$$0 \rightarrow \mathcal{U} \rightarrow \mathcal{V} \rightarrow \mathcal{W} \rightarrow 0 \quad \text{S.E.S. of vector bundles} \implies [\mathcal{V}] = [\mathcal{U}] + [\mathcal{W}].$$

For a line bundle $\mathcal{L}$ on X , define

$$c_1^K(\mathcal{L}) := 1 - [\mathcal{L}^\vee].$$

For line bundles $\mathcal{L}_1, \mathcal{L}_2$ on X , there is a **fundamental relation**:

$$\begin{aligned} c_1^K(\mathcal{L}_1 \otimes_X \mathcal{L}_2) &= 1 - [\mathcal{L}_1^\vee \otimes_X \mathcal{L}_2^\vee] \\ &= (1 - [\mathcal{L}_1^\vee]) + (1 - [\mathcal{L}_2^\vee]) - (1 - [\mathcal{L}_1^\vee])(1 - [\mathcal{L}_2^\vee]) \\ &= F_M(c_1^K(\mathcal{L}_1), c_1^K(\mathcal{L}_2)). \end{aligned}$$

How I think about algebraic cobordism: Part I

The **algebraic cobordism ring** $\Omega^*(X)$ was constructed by Levine and Morel in the early 2000s.

Their construction was formal, based off Quillen's 1969 formalization of complex cobordism as a "complex-oriented cohomology theory".

The ring $\Omega^*(X)$ is **generated** by classes of projective morphisms $[Y \rightarrow X]$, where Y is a smooth, quasi-projective, and irreducible.

The theory Ω^* possesses **Chern classes**. As the inclusion of a smooth divisor $D \hookrightarrow X$ is closed embedding, it defines class $[D \hookrightarrow X] \in \Omega^*(X)$, and

$$c_1^{\Omega^*}(\mathcal{O}_X(D)) = [D \hookrightarrow X].$$

How I think about algebraic cobordism: Part II

The Chern classes satisfy the **Quillen formula**:

$$\begin{aligned}c_1^{\Omega^*}(\mathcal{L}_1 \otimes_X \mathcal{L}_2) &= F_U(c_1^{\Omega^*}(\mathcal{L}_1), c_1^{\Omega^*}(\mathcal{L}_2)) \\ \implies c_1^{\Omega^*}(\mathcal{L}^{\vee}) &= -F_U(c_1^{\Omega^*}(\mathcal{L})).\end{aligned}$$

Based on observations of Milnor and Novikov in the 1960s, $\Omega^*(\text{point})$ is a **geometric realization** of the Lazard ring:

$$\Omega^*(\text{point}) \simeq \mathbb{L}^* \simeq \mathbb{Z}[x_1, x_2, x_3, \dots], \quad \deg(x_i) = 2i.$$

The x_i are built from projective spaces and complex hypersurfaces.

$\Omega^*(X)$ is algebra over $\Omega^*(\text{point})$. There are ring maps $\Omega^*(X) \rightarrow \text{CH}^*(X)$ and $\Omega^*(X) \rightarrow K(X)$ where $\mathbb{L}^*$ maps to $\mathbb{Z}$ based on the group law, and

$$c_1^{\Omega^*}(\mathcal{O}_X(D)) \mapsto c_1^{\text{CH}^*}(\mathcal{O}_X(D)) \quad \text{and} \quad c_1^{\Omega^*}(\mathcal{O}_X(D)) \mapsto c_1^K(\mathcal{O}_X(D)).$$

First computation: complex projective space

CH*: A hyperplane H in $\mathbb{P}^n$ defines a class $c_1^{\text{CH}^*}(\mathcal{O}_{\mathbb{P}^n}(H)) = [H]$, and

$$\text{CH}^*(\mathbb{P}^n) \simeq \mathbb{Z}[H]/(H^{n+1}).$$

K: It also defines a class $c_1^K(\mathcal{O}_{\mathbb{P}^n}(H)) = 1 - [\mathcal{O}_{\mathbb{P}^n}(-H)]$, and

$$K(\mathbb{P}^n) \simeq \mathbb{Z}[c_1^K(\mathcal{O}_{\mathbb{P}^n}(H))]/((c_1^K(\mathcal{O}_{\mathbb{P}^n}(H)))^{n+1}).$$

Observe the "exceptional" ring isomorphism:

$$K(\mathbb{P}^n) \xrightarrow{\simeq} \text{CH}^*(\mathbb{P}^n), \quad c_1^K(\mathcal{O}_{\mathbb{P}^n}(H)) \mapsto c_1^{\text{CH}^*}(\mathcal{O}_{\mathbb{P}^n}(H)).$$

Ω^* : Finally, it defines a class $c_1^{\Omega^*}(\mathcal{O}_{\mathbb{P}^n}(H)) = [H \hookrightarrow \mathbb{P}^n]$, and

$$\begin{aligned} \Omega^*(\mathbb{P}^n) &\simeq \mathbb{L}^*[[H \hookrightarrow \mathbb{P}^n]]/([H \hookrightarrow \mathbb{P}^n]^{n+1}) \\ &\simeq \mathbb{L}^* \otimes_{\mathbb{Z}} \text{CH}^*(\mathbb{P}^n) \quad \textbf{(This line is surprising.)} \end{aligned}$$

This recovers $\text{CH}^*(\mathbb{P}^n)$, $K(\mathbb{P}^n)$, and the exceptional isomorphism.

Part 2:

Wonderful varieties and
exceptional isomorphisms

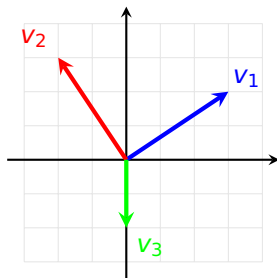
Realizable matroids

Let M be the **matroid** for a set of vectors $E := \{v_1, \dots, v_n\}$ in $\mathbb{C}^r$.

The **independent sets** of M are the linearly independent subsets of E .

The **rank** $r(S)$ of a subset $S \subseteq E$ is the dimension of its span.

A **flat** is a subset $G \subseteq E$ such that $r(G \cup v_i) > r(G)$ for all $v_i \in E \setminus G$.



Independent sets:

$\{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}$

Flats:

$\{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2, v_3\}\}$

Wonderful varieties of hyperplane arrangements

Let $\mathcal{H} := \{H_i\}_{i=1}^n$ be **distinct** hyperplanes in $\mathbb{P}(\mathbb{C}^r) = \mathbb{P}^{r-1}$ with $\bigcap_{i=1}^n H_i = (0)$. Set $U := \mathbb{P}^{r-1} \setminus \bigcup_{i=1}^n H_i$.

The *normal vectors* of the H_i in $\mathbb{C}^r$ define a **matroid** $M_{\mathcal{H}}$.

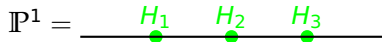
For any **flat** $G \in M_{\mathcal{H}}$, set $L_G := \bigcap_{i \in G} H_i$.

The **wonderful variety** $W_{\mathcal{H}}$ is a smooth SNC compactification of U :

STEP 1: blow up the 0-dimensional L_G 's in $\mathbb{P}^{r-1} \rightsquigarrow W_{\mathcal{H}}^{(1)}$

STEP 2: blow up the strict transforms of the 1-dimensional L_G 's $\rightsquigarrow W_{\mathcal{H}}^{(2)}$

STEP 3: continue this way until reaching $W_{\mathcal{H}} := W_{\mathcal{H}}^{(r)}$.

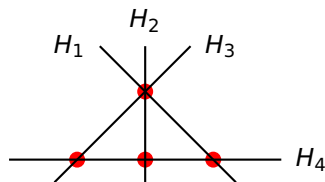


$$\mathcal{H} = \{H_1, H_2, H_3\}$$

Flats: $\{\emptyset, \{H_1\}, \{H_2\}, \{H_3\}, \mathcal{H}\}$

The H_i are also **exceptional divisors**.

One more example: exceptional divisors

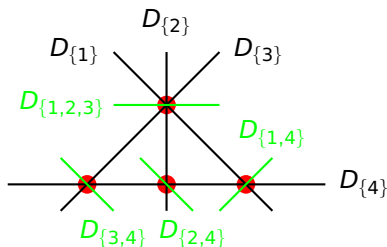


$$\mathcal{H} = \{H_1, H_2, H_3, H_4\} \text{ in } \mathbb{P}^2$$

Flats of $M_{\mathcal{H}}$:

$$\{\emptyset, \{H_1\}, \{H_2\}, \{H_3\}, \{H_4\}, \{H_1, H_4\}, \{H_2, H_4\}, \{H_3, H_4\}, \{H_1, H_2, H_3\}, \mathcal{H}\}$$

$W_{\mathcal{H}} =$



The D_G are
**exceptional
divisors**
(indexed by
proper,
non-empty flats of
 $M_{\mathcal{H}}$).

The Chow ring of $W_{\mathcal{H}}$

(1995) The variety $W_{\mathcal{H}}$ was constructed by De Concini–Procesi.

(2004) Feichtner-Yuzvinsky gave a simple presentation for $\mathrm{CH}^*(W_{\mathcal{H}})$, **in terms of $M_{\mathcal{H}}$** :

Exceptional divisors $[D_G]$ (G proper, nonempty) **generate** $\mathrm{CH}^*(W_{\mathcal{H}})$.

Flats G and H are **incomparable** if $G \not\subseteq H$ and $H \not\subseteq G$. Here, $D_G \cap D_H = \emptyset$.

Fix $H_i \in \mathcal{H}$, and set $[D_{\mathcal{H}}] := \sum_{H_i \in G \neq \mathcal{H}} (-[D_G])$ (**Independent of H_i !**).

$$\mathrm{CH}^*(W_{\mathcal{H}}) \xrightarrow{\sim} \frac{\mathbb{Z}[x_G]_{\{G \text{ non-empty flat}\}}}{I+J}, \quad [D_G] \mapsto x_G$$

$$I = \langle x_G x_H \mid G, H \text{ incomparable} \rangle \quad \text{and} \quad J = \left\langle \sum_{H_i \in G} x_G \mid H_i \in \mathcal{H} \right\rangle.$$

Repackaging using Chern classes

As $W_{\mathcal{H}}$ is smooth, recall the **group isomorphism**

$$c_1^{\text{CH}^*} : \text{Pic}(W_{\mathcal{H}}) \xrightarrow{\sim} \text{Div}(W_{\mathcal{H}}) / \sim, \quad \mathcal{L} = \mathcal{O}_X(D) \mapsto c_1^{\text{CH}^*}(\mathcal{O}_X(D)) = [D].$$

For $H_i \in \mathcal{H}$, set $\mathcal{O}_{W_{\mathcal{H}}}(D_{\mathcal{H}}) := \bigotimes_{H_i \in G \neq \mathcal{H}} \mathcal{O}_{W_{\mathcal{H}}}(-D_G)$. Then

$$\sum_{H_i \in G} c_1^{\text{CH}^*}(\mathcal{O}_{W_{\mathcal{H}}}(D_G)) = 0 \iff \bigotimes_{H_i \in G} \mathcal{O}_{W_{\mathcal{H}}}(D_G) = \mathcal{O}_{W_{\mathcal{H}}}$$

Therefore, the relation J is **equivalent to** a relation in $\text{Pic}(X)$.

By this logic, for **any** oriented cohomology theory h^* with group law F :

$$0 = c_1^{h^*}(\mathcal{O}_{W_{\mathcal{H}}}) = c_1^{h^*} \left(\bigotimes_{H_i \in G} \mathcal{O}(D_G) \right) = \sum_{H_i \in G}^F c_1^{h^*}(\mathcal{O}(D_G)).$$

General Feichter-Yuzvinsky presentations

(2024) Larson–Li–Payne–Proudfoot gave the following **FY presentation**:

$$K(W_{\mathcal{H}}) \simeq \frac{\mathbb{Z}[x_G]_{\{G \text{ non-empty flat}\}}}{\langle \mathbf{1} - \prod_{H_i \in \mathcal{H}} (1 - x_G) \mid H_i \in \mathbf{G} \rangle + \langle x_G x_H \mid G, H \text{ incomparable} \rangle}$$

Here: $x_G = c_1^K(\mathcal{O}(D_G)) = 1 - [\mathcal{O}(D_G)^\vee]$ for $G \neq \mathcal{H}$.

Recall:

$$1 - \prod_{i=1}^n (1 - t_i) = t_1 +_{F_M} t_2 +_{F_M} \cdots +_{F_M} t_n.$$

Thus, the **red** relation is equivalent to $\sum_{H_i \in \mathbf{G}}^{F_M} x_G = 0$.

Proposition (G.-Partida 2026)

With $x_G := c_1^{\Omega^*}(\mathcal{O}(D_G)) = [D_G \hookrightarrow W_{\mathcal{H}}]$, there is $\mathbb{L}^*$ -algebra isomorphism

$$\Omega^*(W_{\mathcal{H}}) \simeq \frac{\mathbb{L}^*[x_G]_{\{G \text{ non-empty flat}\}}}{\langle \sum_{H_i \in \mathbf{G}}^{F_U} x_G \mid H_i \in \mathcal{H} \rangle + \langle x_G x_H \mid G, H \text{ incomparable} \rangle}.$$

Simplicial presentation

(2023) Backman–Eur–Simpson gave **new presentation** of $\mathrm{CH}^*(W_{\mathcal{H}})$:

The **simplicial generators** are:

$$\mathcal{L}_F := \bigotimes_{F \subseteq G} \left(\mathcal{O}(D_G)^\vee \right) \quad \text{for non-empty } F.$$

$$\text{Möbius inversion} \implies \mathcal{O}(D_F)^\vee = \bigotimes_{F \subseteq G} \mathcal{L}_G^{\otimes \mu(F,G)}.$$

With the notation $x_F := c_1^{\mathrm{CH}^*}(\mathcal{O}(D_F))$ and $h_F := c_1^{\mathrm{CH}^*}(\mathcal{L}_F)$:

$$h_F = \sum_{F \subseteq G} (-x_G) \quad \text{and} \quad -x_F = \sum_{F \subseteq G} (\mu(F, G) \cdot h_G).$$

The **simplicial presentation** is

$$\mathrm{CH}^*(W_{\mathcal{H}}) \simeq \frac{\mathbb{Z}[h_G]_{\{G \text{ non-empty flat}\}}}{\langle (h_F - h_{F \vee G})(h_G - h_{F \vee G}) \mid G, F \text{ non-empty} \rangle + \langle h_F \mid \mathrm{rank}(F) = 1 \rangle}$$

Back to general group laws

For **any** oriented cohomology ring $h^*(W_{\mathcal{H}})$ with group law F , set

$$x_G := c_1^{h^*}(\mathcal{O}(D_G)) \quad \text{and} \quad h_G := c_1^{h^*}(\mathcal{L}_G).$$

The relations in $\text{Pic}(X)$ imply:

$$\mathcal{L}_G = \bigotimes_{G \subseteq H} \left(\mathcal{O}(D_H)^\vee \right) \implies h_G = \sum_{G \subseteq H}^F (-_F X_H).$$

$$\mathcal{O}(D_G)^\vee = \bigotimes_{G \subseteq H} \mathcal{L}_H^{\otimes \mu(G,H)} \implies (-_F X_G) = \sum_{G \subseteq H}^F (\mu(G,H) \cdot_F h_H).$$

The exceptional isomorphism

(2024) Larson-Li-Payne-Proudfoot gave a **simplicial presentation**:

$$K(W_{\mathcal{H}}) \xrightarrow{\cong} \frac{\mathbb{Z}[h_G]_{\{G \text{ non-empty flat}\}}}{\langle (h_F - h_{F \vee G})(h_G - h_{F \vee G}) \rangle + \langle h_F \mid \text{rank}(F) = 1 \rangle},$$

where $x_G \mapsto \sum_{G \subseteq H} {}^{F_M}(-\mu(G, H) \cdot {}_{F_M} h_H)$.

Remarkably, this presentation is **identical** to that of $\text{CH}^*(W_{\mathcal{H}})$:

$$K(W_{\mathcal{H}}) \xrightarrow{\cong} \text{CH}^*(W_{\mathcal{H}}), \quad h_G \mapsto h_G.$$

This is the **exceptional isomorphism** (BEST (2023) and LLPP (2024)).

It is **not that same as** the Chern character isomorphism

$$\text{ch}: K(W_{\mathcal{H}}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \text{CH}^*(W_{\mathcal{H}}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The cobordism exceptional isomorphism

Recall the Chow ring **simplicial presentation**:

$$\mathrm{CH}^*(W_{\mathcal{H}}) \simeq \frac{\mathbb{Z}[h_G]_{\{G \text{ non-empty flat}\}}}{\langle (h_F - h_{F \vee G})(h_G - h_{F \vee G}) \rangle + \langle h_F \mid \mathrm{rank}(F) = 1 \rangle}$$

Theorem (G.-Partida 2026)

There is an $\mathbb{L}^*$ -algebra isomorphism:

$$\Omega^*(W_{\mathcal{H}}) \xrightarrow{\sim} \frac{\mathbb{L}^*[h_G]_{\{G \text{ non-empty flat}\}}}{\langle (h_F - h_{F \vee G})(h_G - h_{F \vee G}) \rangle + \langle h_F \mid \mathrm{rank}(F) = 1 \rangle},$$

sending $x_G \mapsto \sum_{G \subseteq H}^{F_U} ((-\mu(G, H)) \cdot_{F_U} h_H)$.

Consequently, there is an **exceptional** $\mathbb{L}^*$ -algebra isomorphism:

$$\Omega^*(W_{\mathcal{H}}) \xrightarrow{\sim} \mathbb{L}^* \otimes_{\mathbb{Z}} \mathrm{CH}^*(W_{\mathcal{H}}), \quad h_G \mapsto 1 \otimes h_G.$$

Part 3:

Other results and what's next?

Non-realizable matroids

(2004) Feichtner–Yuzvinsky showed there is a toric variety $X_{M_{\mathcal{H}}}$ such that:

$$\iota: W_{\mathcal{H}} \hookrightarrow X_{M_{\mathcal{H}}} \quad \text{and} \quad \iota^*: \mathrm{CH}^*(X_{M_{\mathcal{H}}}) \xrightarrow{\cong} \mathrm{CH}^*(W_{\mathcal{H}}).$$

The toric variety $X_{M_{\mathcal{H}}}$ is often **non-compact**.

The construction of X_M works for any loopless matroid M , and X_M was used to prove (by AHK in 2018) the **Heron–Rota–Welsch conjecture**.

BEST (2023) and LLPP (2024) showed:

$$\mathrm{CH}^*(X_M) \xrightarrow{\cong} K(X_M), \quad h_G \mapsto h_G.$$

(2023) Speyer: there is a **smooth, projective** toric variety X with

$$\mathrm{CH}^*(X) \not\cong K(X).$$

Theorem (G.–Partida (2026))

For any loopless matroid M : $\Omega^*(X_M) \xrightarrow{\cong} \mathbb{L}^* \otimes_{\mathbb{Z}} \mathrm{CH}^*(X_M), \quad h_G \mapsto 1 \otimes h_G.$

Cobordism class of a wonderful variety

The ring $\Omega^*(\text{point})$ is generated by classes of morphisms $[Y \rightarrow \text{point}]$, where Y is smooth, irreducible, and projective.

(1971) Quillen: for X_1 and X_2 smooth, projective varieties, the classes $[X_1 \rightarrow \text{point}]$ and $[X_2 \rightarrow \text{point}]$ are equal if and only if all the Chern numbers of X_1 and X_2 are equal.

(2026) Cheng: the Chern numbers of $W_{\mathcal{H}}$ depend only on $M_{\mathcal{H}}$.

Corollary (G.-Partida (2026))

The class $[W_{\mathcal{H}} \rightarrow \text{point}]$ depends only on $M_{\mathcal{H}}$.

Work in progress:

Recursive formula for $[W_{\mathcal{H}} \rightarrow \text{point}]$ in terms of generators of $\Omega^*(\text{point})$.

The exceptional Hirzebruch-Riemann-Roch

The K -theoretic exceptional isomorphism:

$$\zeta: K(W_{\mathcal{H}}) \xrightarrow{\cong} \mathrm{CH}^*(W_{\mathcal{H}})$$

satisfies an **exceptional Hirzebruch-Riemann-Roch** formula:

$$\chi(x) = \deg(\zeta(x) \cdot \mathrm{Td}), \quad x \in K(X).$$

Here, $\mathrm{Td} = \frac{1}{1+c_1(\mathcal{O}(D_{\mathcal{H}}))}$ is the **exceptional Todd class**.

Work in progress:

Determine the **cobordism analogue** of the exceptional Hirzebruch-Riemann-Roch arising from the exceptional isomorphism

$$\Omega^*(W_{\mathcal{H}}) \simeq \mathbb{L}^* \otimes_{\mathbb{Z}} \mathrm{CH}^*(W_{\mathcal{H}}).$$

Thank you!