Deforming the motivic Segre classes of Schubert cells in the Grassmannian

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Divided difference operators

Define $R := \mathbb{Z}[x_1, \dots, x_n]$. Let s_i be the transposition in S_n that swaps i and $i+1$. This defines an action of S_n on R, where s_i swaps x_i and x_{i+1}.

Definition (Demazure 1973, 1974)

Consider the $\mathbb{Z}$-linear operators on R, one for each $i = 1, \dots, n-1$:

$$\partial_i(f) := \frac{f - s_i(f)}{x_{i+1} - x_i}, \quad f \in R.$$

The ∂_i are called **divided difference operators**.

For $w = s_{i_1} \circ \cdots s_{i_k}$ reduced, define $\partial_w := \partial_{s_{i_1}} \circ \cdots \circ \partial_{s_{i_k}}$. The operator ∂_w does not depend on the choice of reduced expression for w.

Example

$$\partial_2(x_1 x_3) = \frac{x_1 x_3 - s_2(x_1 x_3)}{x_3 - x_2} = \frac{x_1 x_3 - x_1 x_2}{x_3 - x_2} = x_1 \in R.$$

S_n-actions

Let Λ_k^n be the set of 01 sequences with k 1's and $n-k$ 0's. The swap s_i acts on Λ_k^n by swapping the i-th and $(i+1)$-th entries of a sequence. Define the word $\omega := 1^k 0^{n-k}$.

Example

The sequence $1001110 \in \Lambda_4^7$. We have $s_2(1001110) = 0101110$.

Consider the ring $\widetilde{R} := \bigoplus_{\Lambda_k^n} R = \bigoplus_{\Lambda_k^n} \mathbb{Z}[x_1, \ldots, x_n]$.

The transposition s_i acts on $\widetilde{R}$ by $s_i((f_\lambda)_{\lambda \in \Lambda_k^n}) := (s_i(f_\lambda))_{s_i(\lambda) \in \Lambda_k^n}$.

Example

Consider $(f_{110}, f_{101}, f_{011}) = (x_1x_2, x_2^2, x_1x_3^4) \in \widetilde{R}$, indexed by Λ_2^3. Then

$$s_1(x_1x_2, x_2^2, x_1x_3^4) = (s_1(x_1x_2), s_1(x_1x_3^4), s_1(x_2^2)) = (x_1x_2, x_2x_3^4, x_1^2).$$

GKM conditions and Schubert classes

Definition (Goresky-Kottwitz-MacPherson 1996)

An element $(f_\lambda)_{\lambda \in \Lambda_k^n} \in \widetilde{R}$ is called **GKM** if:

whenever $\lambda = (i,j)(\lambda')$, the difference $f_\lambda - f_{\lambda'}$ is divisible by $x_i - x_j$ in R.

Example

The sequences $(1,1,1)$ and $(0,0,0,(x_1-x_2)(x_1-x_3)(x_2-x_3))$ in $\widetilde{R}$ indexed by $\Lambda_1^3 = \{(f_{100}, f_{010}, f_{001})\}$ are GKM.

Definition (Schubert classes)

Fix Λ_k^n. Define in $\widetilde{R}$, an element $S_\omega|_\lambda := \begin{cases} \prod_{i>j:\lambda_i<\lambda_j} x_i - x_j, & \text{if } \lambda = \omega; \\ 0, & \text{otherwise.} \end{cases}$

The other S_λ are defined by the rule $S_{w^{-1}(\omega)} := \partial_w(S_\omega)$.

The S_λ are GKM and called **Schubert classes**.

Multiplying Schubert classes

Definition (Schubert basis)

The $\mathbb{Z}[x_1, \dots, x_n]$-subalgebra of $\widetilde{R}$ generated by $\{S_\lambda\}_{\lambda \in \Lambda_k^n}$ is $H_T(\mathrm{Gr}(k,n))$.
The S_λ form $\mathbb{Z}[x_1, \dots, x_n]$-basis for the subalgebra: the **Schubert basis**.

Let us run an example for Λ_1^2. Recall the operator

$$\partial_1(f) := \frac{f - s_1(f)}{x_2 - x_1}.$$

We have

$$S_{10} = [0, x_2 - x_1]\,; \qquad S_{01} = \partial_1(S_{10}) = [1,1]\,.$$

Let us compute all products and express them in terms of the S_λ:

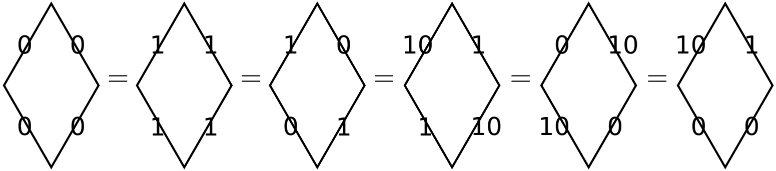
$$S_{10}^2 = (x_2 - x_1)S_{10}; \quad S_{10} \cdot S_{01} = S_{10}; \quad S_{01}^2 = S_{01}.$$

The structure constants lie in $\mathbb{N}[x_2 - x_1]$.

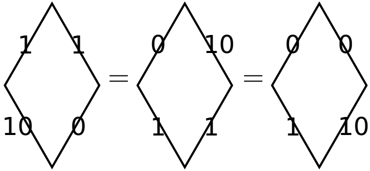
Question

Is there a combinatorial formula for the structure constants in S_λ basis?

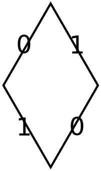
Consider the following **puzzle pieces**, equipped with a function from $\{1,2,3,\dots\}^2$ to $\mathbb{Z}[x_1,x_2,\dots]$ called its **fugacity**.



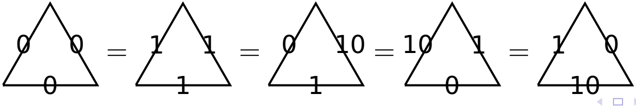
$= 1$



$= 1$



$= x_j - x_i.$

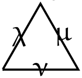


$= 1$ (allow rotations)

Knutson-Tao puzzles

A **Knutson-Tao puzzle** is a triangle with side labels λ, μ, ν in Λ_k^n that is tiled by the puzzle pieces.

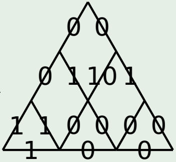
The **fugacity** of a puzzle is the product of fugacities of its tiles. The fugacity of a rhombus tile is $x_i - x_j$, where i is the i-th NE-to-SW diagonal, and j is the j-th NW-to-SE diagonal in the puzzle.



$:=$ sum of puzzle fugacities over all puzzles with λ, μ, ν boundary.

Example

For $\lambda = 100$ (left), $\mu = 010$ (right), $\nu = 100$ (bottom):

fug 

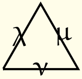
$= (x_1 - x_2) \cdot 1 \cdot 1 = x_1 - x_2.$

Knutson-Tao puzzles

Theorem (Knutson-Tao 2003)

For any $\lambda, \mu \in \Lambda_k^n$, the product $S_\lambda \cdot S_\mu$ is

$$S_\lambda \cdot S_\mu = \sum_\nu$$



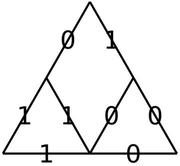
S_ν.

Thus the structure constants lie in $\mathbb{N}[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]$.

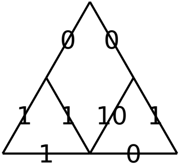
Recall our computation in a previous example:

$$S_{10}^2 = (x_1 - x_2)S_{10}; \quad S_{10} \cdot S_{01} = S_{10}; \quad S_{01}^2 = S_{01}.$$

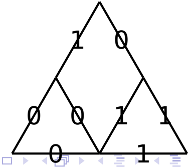
We compute



$= x_1 - x_2$



$= 1$



$= 1$

Positive formulas

Question

What is a positive formula?

Example

Say I have a basis $B_1, \ldots, B_n$, and the structure constants for this basis live in $\mathbb{N}$. The structure constants are **positive** because $\mathbb{N}$ is a monoid and $\mathbb{N} \cap (-\mathbb{N}) = (0)$.

Definition (Knutson–Zinn-Justin 2021)

A **positivity monoid** is a monoid M such that $M \cap (-M) = (0)$. If the structure constants for a basis live in a positivity monoid, then the structure constants are **positive**.

Example

$\mathbb{N}[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]$ is a positivity monoid.

K-theory divided difference operator

Define $R := \mathbb{Z}[e^{\pm x_1}, \ldots, e^{\pm x_n}]$. Let s_i be the transposition in S_n that swaps i and $i+1$. This defines an action of S_n on R, where s_i swaps e^{x_i} and $e^{x_{i+1}}$.

Definition (Demazure 1973, 1974)

Consider the $\mathbb{Z}$-linear operators on R, one for each $i = 1, \ldots, n-1$:

$$\partial_i(f) := \frac{f - e^{x_{i+1}-x_i} s_i(f)}{1 - e^{x_{i+1}-x_i}}, \quad f \in R.$$

The ∂_i are called **divided difference operators**.
For $w = s_{i_1} \circ \cdots s_{i_k}$ reduced, define $\partial_w := \partial_{s_{i_1}} \circ \cdots \circ \partial_{s_{i_k}}$. The ∂_w does not depend on the choice of reduced expression for w.

Example

$$\partial_1(e^{x_1}) = \frac{e^{x_1} - e^{x_2-x_1} s_1(e^{x_1})}{1 - e^{x_2-x_1}} = \frac{e^{x_1}(1 - e^{2x_2-2x_1})}{1 - e^{x_2-x_1}} = \frac{e^{x_1}(1 - e^{x_2-x_1})(1 + e^{x_2-x_1})}{1 - e^{x_2-x_1}} \in R.$$

K-theory GKM conditions

Define the ring $\widetilde{R} := \bigoplus_{\Lambda_k^n} R$. Recall the action $s_i((f_\lambda)_\lambda) := (s_i(f_\lambda))_{s_i(\lambda)}$.

Definition (e.g., Knutson-Roşu, Cor. A.5, 2003)

An element $(f_\lambda)_{\lambda \in \Lambda_k^n} \in \widetilde{R}$ is called **GKM** if:

whenever $\lambda = (i,j)(\lambda')$, we have $f_\lambda - f_{\lambda'}$ is divisible by $1 - e^{x_i - x_j}$ in R.

Definition (Schubert classes)

Fix Λ_k^n. Define in $\widetilde{R}$, an element

$$S_\omega|_\lambda := \begin{cases} \prod_{i>j:\lambda_i<\lambda_j}(1-e^{x_i-x_j}), & \text{if } \lambda = \omega; \\ 0, & \text{otherwise.} \end{cases}$$

The other S_λ are defined by the rule $S_{w^{-1}(\omega)} := \partial_w(S_\omega)$.
The S_λ are GKM and called **Schubert classes**.

Multiplying Schubert classes

Definition (Schubert basis)

The $\mathbb{Z}[e^{\pm x_1}, \ldots, e^{\pm x_n}]$-subalgebra of $\widetilde{R}$ generated by $\{S_\lambda\}_{\lambda \in \Lambda_k^n}$ is $K_T(\mathrm{Gr}(k,n))$. The S_λ form $\mathbb{Z}[e^{\pm x_1}, \ldots, e^{\pm_n}]$-basis for the subalgebra: the **Schubert basis**.

Theorem (Pechenik-Yong 2017, Wheeler-Zinn-Justin 2019)

For any $\lambda, \mu \in \Lambda_k^n$, the product $S_\lambda \cdot S_\mu$ is

$$S_\lambda \cdot S_\mu = \sum_\nu \overset{\lambda\ \ \mu}{\underset{\nu}{\triangle}} \, S_\nu,$$

where the tiles and fugacities of puzzle pieces are now different. The structure constants are "positive", in the sense: $(-1)^{\ell(\nu)-\ell(\lambda)-\ell(\mu)} \overset{\lambda\ \mu}{\underset{\nu}{\triangle}}$ lies in the positivity monoid

$$\mathbb{N}[e^{x_2-x_1}, e^{x_3-x_2}, \ldots, e^{x_n-x_{n-1}}, 1-e^{x_2-x_1}, 1-e^{x_3-x_2}, \ldots, 1-e^{x_n-x_{n-1}}].$$

$\hbar$-deformations of H_T classes

Define $R := \mathbb{Z}[x_1, \ldots, x_n, \hbar]$. Define an action of S_n on R, where s_i swaps x_i and x_{i+1} and fixes $\hbar$. Define the ring $\widetilde{R} := \oplus_{\lambda \in \Lambda_k^n} \mathrm{Frac}(R)$.

Definition

Consider the $\mathbb{Z}$-linear operators on R, one for each $i = 1, \ldots, n-1$:

$$\partial_i := \frac{\hbar}{x_i - x_{i+1}} + \frac{x_i - x_{i+1} - \hbar}{x_i - x_{i+1}} s_i.$$

The ∂_i will be called "**cohomological Deligne-Lusztig operators**".

Define in $\widetilde{R}$, an element $S_\omega|_\lambda := \begin{cases} \prod_{i>j:\lambda_i<\lambda_j} \frac{x_i - x_j}{\hbar - (x_i - x_j)}, & \text{if } \lambda = \omega; \\ 0, & \text{otherwise.} \end{cases}$

The other S_λ are defined by the rule $S_{w^{-1}(\omega)} := \partial_w(S_\omega)$.

The S_λ are called **Segre-Schwartz-MacPherson classes**.

There is a positive puzzle formula for the structure constants for S_λ in terms of Knutson-Tao puzzles [Knutson–Zinn-Justin 2021].

q-deformation of K_T classes

Define $R := \mathbb{Z}[e^{\pm x_1}, \dots, e^{\pm x_n}, q^2]$. Define an action of S_n on R, where s_i swaps e^{x_i} and $e^{x_{i+1}}$ and fixes q^2. Define the ring $\widetilde{R} := \oplus_{\lambda \in \Lambda_k^n} \mathrm{Frac}(R)$.

Definition

Consider the $\mathbb{Z}$-linear operators on R, one for each $i = 1, \dots, n-1$:

$$\partial_i := \frac{1-q^2}{1-e^{x_{i+1}-x_i}} + \frac{1-q^2 e^{x_i - x_{i+1}}}{1-e^{x_i - x_{i+1}}} s_i.$$

The ∂_i are called **Deligne-Lusztig operators**.

Define in $\widetilde{R}$, an element $S_\omega|_\lambda := \begin{cases} \prod_{i>j:\lambda_i<\lambda_j} \frac{1-e^{x_j-x_i}}{1-q^2 e^{x_j-x_i}}, & \text{if } \lambda = \omega; \\ 0, & \text{otherwise.} \end{cases}$

The other S_λ are defined by the rule $S_{w^{-1}(\omega)} := \partial_w(S_\omega)$.

The S_λ are called **motivic Segre classes**.

There is a positive puzzle formula for the structure constants for S_λ in terms of Knutson-Tao puzzles [Knutson–Zinn-Justin 2021].

A note on Chern classes

Remark

The element $1 - e^{x_i - x_{i+1}}$ is the first equivariant Chern class (in K-theory) of the homogeneous line bundle $\mathcal{L}_{x_{i+1}-x_i} \to G/B$. Let's replace $1 - e^{x_i - x_{i+1}}$ by $c_1(\mathcal{L}_{x_{i+1}-x_i})$ everywhere in the motivic Segre classes.

$$K_T : \qquad \partial_i := \frac{1-q^2}{c_1(\mathcal{L}_{x_i - x_{i+1}})} + \frac{1-q^2(1-c_1(\mathcal{L}_{x_i-x_{i+1}}))}{c_1(\mathcal{L}_{x_{i+1}-x_i})} s_i.$$

$$S_\omega|_\lambda := \begin{cases} \prod_{i>j:\lambda_i<\lambda_j} \frac{c_1(\mathcal{L}_{x_i-x_j})}{1-q^2(1-c_1(\mathcal{L}_{x_i-x_j}))}, & \text{if } \lambda = \omega; \\ 0, & \text{otherwise.} \end{cases}$$

$$S_{w^{-1}(\omega)} := \partial_w(S_\omega).$$

Question

What if we replace c_1 by a Chern class in another cohomology theory?

'Connective' K-theory

An algebraic oriented cohomology theory h^* is a functor:

$$\mathrm{h}^* : \{\text{smooth algebraic varieties}\} \to \{\text{graded, commutative, unital rings}\},$$

that satisfies 'cohomology-type' axioms.

Example

Chow ring theory and K-theory are oriented cohomology theories.

There is an oriented cohomology theory called **connective K-theory**. After a localization, the first equivariant Chern class in connective K-theory sends $\mathcal{L}_{x_{i+1}-x_i}$ to $\beta^{-1}(1-e^{x_i-x_{i+1}})$, where β is a free variable.

Let's replace everything with this new Chern class!

Deforming the motivic Segre classes

The new operator and classes for connective K-theory (after localizing):

$$\partial_i := \frac{\beta(1-q^2)}{1-e^{x_{i+1}-x_i}} + \frac{\beta(1-q^2)+q^2(1-e^{x_i-x_{i+1}})}{1-e^{x_i-x_{i+1}}} s_i.$$

$$S_\omega|_\lambda := \begin{cases} \prod_{i>j:\lambda_i<\lambda_j} \frac{1-e^{x_i-x_j}}{\beta(1-q^2)+q^2(1-e^{x_i-x_j})}, & \text{if } \lambda=\omega; \\ 0, & \text{otherwise.} \end{cases}$$

$$S_{w^{-1}(\omega)} := \partial_w(S_\omega)$$

Lemma

$\partial_w := \partial_{i_1} \circ \cdots \circ \partial_{i_k}$ is independent of the reduced expression $w = s_{i_1} \cdots s_{i_k}$:

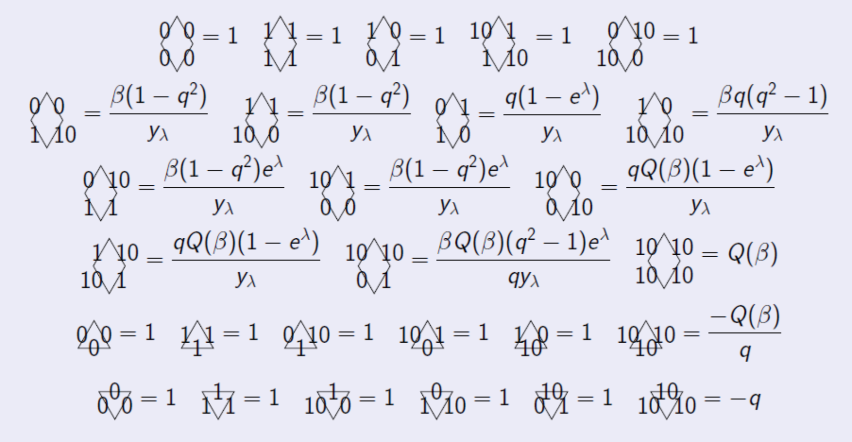
1. $\partial_i \circ \partial_{i+1} \circ \partial_i = \partial_{i+1} \circ \partial_i \circ \partial_{i+1}$ for $i = 1, \ldots, n-2$.
2. $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ for all $|i-j| > 1$.

Therefore, the classes S_λ are well-defined.

The $\beta = 1$ specialization recovers the motivic Segre classes $S_\lambda^{K_T}$.

The $\beta = 0$ 'limit' recovers the homogenizations $(\hbar+1)^{\mathrm{length}(\lambda)} S_\lambda^{H_T}$.

The puzzle formula



Theorem (G. 2025+)

$$\left(q^{\mathrm{length}(\lambda)} S_\lambda\right) \cdot \left(q^{\mathrm{length}(\mu)} S_\mu\right) = \sum\nolimits_\nu \overset{\lambda\quad\mu}{\underset{\nu}{\triangle}} \left(q^{\mathrm{length}(\nu)} S_\nu\right)$$

Positivity

Define $Q(\beta) := q^2 + \beta - q^2\beta$.

Consider the submonoid M of $\mathrm{Frac}(\mathbb{Z}[\beta][e^{\pm x_1}, \ldots, e^{\pm x_n}, q^{\pm 1}])$, defined as the set of sums of products of the factors over all $1 \leqslant i < j \leqslant n$:

$$-q^{\pm} \qquad Q(\beta) \qquad e^{x_j - x_i} \qquad \frac{\beta(1-q^2)}{\beta(1-q^2)+q^2(1-e^{x_j-x_i})} \qquad -\frac{1-e^{x_j-x_i}}{\beta(1-q^2)+q^2(1-e^{x_j-x_i})}.$$

Then M is a positivity monoid.

As the structure constants in the S_λ basis live in M, it is in this sense that our puzzle formula is positive.

Question

What *are* the deformed classes S_λ?

Theorem (Localization package)

Let X be a smooth complex algebraic variety that has an algebraic action of a complex torus $T := (\mathbb{C}^\times)^n$, and assume this action has finitely many fixed points F. The natural ring homomorphisms

$$H_T(X) \to \bigoplus_{f \in F} H_T(\mathrm{pt}) \simeq \bigoplus_{f \in F} \mathbb{Z}[x_1, \dots, x_n];$$

$$K_T(X) \to \bigoplus_{f \in F} K_T(\mathrm{pt}) \simeq \bigoplus_{f \in F} \mathbb{Z}[e^{\pm x_1}, \dots, e^{\pm x_n}],$$

induced by the inclusions $\{\text{fixed point}\} \hookrightarrow X$, are injective.

Definition

The **Grassmannian** $\mathrm{Gr}(k, n)$ is the smooth projective algebraic variety consisting of k-dimensional subspaces of $\mathbb{C}^n$. It has an algebraic action of an n-dimensional torus $T := (\mathbb{C}^\times)^n$. The cotangent bundle $T^*(\mathrm{Gr}(k, n))$ has an action of $T \times \mathbb{C}^\times$, where T acts on the base $\mathrm{Gr}(k, n)$ and $\mathbb{C}^\times$ scales the cotangent fibres.

Recall the GKM conditions

Definition

An element $(f_\lambda)_{\lambda\in\Lambda_k^n} \in \bigoplus_{\lambda\in\Lambda_k^n} \mathbb{Z}[x_1,\dots,x_n,\hbar]$ is called **GKM** if:

whenever $\lambda = (i,j)(\lambda')$, the difference $f_\lambda - f_{\lambda'}$ is divisible by $x_i - x_j$.

A GKM class $(f_\lambda)_{\lambda\in\Lambda_k^n}$ can be identified with a class in $H_{T\times\mathbb{C}^\times}(T^*\mathrm{Gr}(k,n))$.

Definition

An element $(f_\lambda)_{\lambda\in\Lambda_k^n} \in \bigoplus_{\lambda\in\Lambda_k^n} K_{T\times\mathbb{C}^\times}(\mathrm{pt}) = \bigoplus_{\lambda\in\Lambda_k^n} \mathbb{Z}[e^{\pm x_1},\dots,e^{\pm x_n},q^2]$ is called **GKM** if:

whenever $\lambda = (i,j)(\lambda')$, we have $f_\lambda - f_{\lambda'}$ is divisible by $1 - e^{x_i - x_j}$.

A GKM class $(f_\lambda)_{\lambda\in\Lambda_k^n}$ can be identified with a class in $K_{T\times\mathbb{C}^\times}(T^*\mathrm{Gr}(k,n))$.

SSM and motivic Segre classes are quotients of classes that satisfy GKM called 'stable classes'.

What *are* the deformed classes?

Recall the operator $\partial_i := \frac{\beta(1-q^2)}{1-e^{x_{i+1}-x_i}} + \frac{\beta(1-q^2)+q^2(1-e^{x_i-x_{i+1}})}{1-e^{x_i-x_{i+1}}} s_i$.
Clear the denominators in the S_λ to define classes St_λ:

$$\mathrm{St}_\omega := \left(\prod_{i>j:\omega_i<\omega_j} (\beta(1-q^2) + q^2(1-e^{x_i-x_j})) \right) S_\omega; \quad \mathrm{St}_{w^{-1}(\omega)} := \partial_w(\mathrm{St}_\omega).$$

Lemma

The elements St_λ satisfy:

whenever $\lambda = (i,j)(\lambda')$, the difference $\mathrm{St}_\lambda - \mathrm{St}_{\lambda'}$ is divisible by $c_1(\mathcal{L}_{x_i-x_j})$.

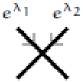
Question (WORK IN PROGRESS)

Does the previous lemma imply that the St_λ come from geometric 'stable classes' in the connective K-ring of $T^*(\mathrm{Gr}(k,n))$?

Answer: Almost surely yes– work in progress

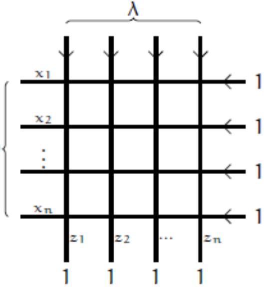
Rational function representatives for deformed classes

$$\widehat{R}(\beta, e^{\lambda})_K :=$$



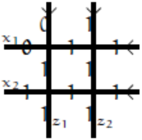
$$= \begin{array}{c} \\ 1\backslash\!/1 \\ 1\backslash\!/0 \\ 0\backslash\!/1 \\ 0\backslash\!/0 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} 1/\!\backslash 1 & 1/\!\backslash 0 & 0/\!\backslash 1 & 0/\!\backslash 0 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\beta(1-q^2)e^{\lambda}}{Q(\beta)-q^2e^{\lambda}} & \frac{qQ(\beta)(1-e^{\lambda})}{Q(\beta)-q^2e^{\lambda}} & 0 \\ 0 & \frac{q(1-e^{\lambda})}{Q(\beta)-q^2e^{\lambda}} & \frac{\beta(1-q^2)}{Q(\beta)-q^2e^{\lambda}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

$$S_\lambda := \omega$$

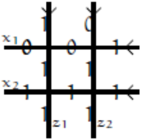


"Sum over all possible grids, and add the fugacities together"

Rational function representatives for deformed classes

$$S_{01} =$$ 

$$= \frac{\beta(1-q^2)}{Q(\beta) - q^2(x_1/z_1)}$$

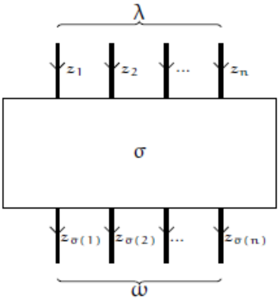
$$S_{10} =$$ 

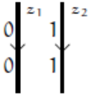
$$= \frac{q(1-x_1/z_1)}{Q(\beta) - q^2(x_1/z_1)} \cdot \frac{\beta(1-q^2)}{Q(\beta) - q^2(x_1/z_2)}$$

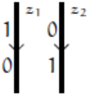
The rational functions S_λ represent the homogenizations $q^{\mathrm{length}(\lambda)} S_\lambda$ of the connective elements S_λ defined earlier.

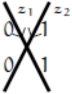
Rational function representatives for deformed classes

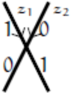
The following diagram equals the evaluation $x_i := z_{\sigma^{-1}(i)}$ in S_λ:

$S_\lambda|_\sigma =$ 

$S_{01}|_{01} =$  $= 1$

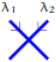
$S_{10}|_{01} =$  $= 0$

$S_{01}|_{10} =$  $= \dfrac{\beta(1-q^2)}{Q(\beta) - q^2(z_2/z_1)}$

$S_{10}|_{10} =$  $= \dfrac{q(1-z_2/z_1)}{Q(\beta) - q^2(z_2/z_1)}$

Proof of puzzle rule: rational function R-matrix

The rational functions S_λ can also be defined using the following matrix entries, with $x_\lambda = \beta^{-1}(1-e^\lambda)$ and $y_\lambda = \beta(1-q^2) + q^2(1-e^\lambda)$.

$R_{bb}(\beta, x_\lambda) =$  $=$

$$
\begin{array}{c|ccccccccc}
 & 1/\backslash 1 & 1/\backslash 0 & 1/\backslash 10 & 0/\backslash 1 & 0/\backslash 0 & 0/\backslash 10 & 10/\backslash 1 & 10/\backslash 0 & 10/\backslash 10 \\
\hline
1\backslash/1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1\backslash/0 & 0 & \frac{(1-q^2)(1-\beta x_\lambda)}{y_\lambda} & 0 & \frac{Q(\beta)qx_\lambda}{y_\lambda} & 0 & 0 & 0 & 0 & 0 \\
1\backslash/10 & 0 & 0 & \frac{1-q^2}{y_\lambda} & 0 & 0 & 0 & \frac{Q(\beta)qx_\lambda}{y_\lambda} & 0 & 0 \\
0\backslash/1 & 0 & \frac{qx_\lambda}{y_\lambda} & 0 & \frac{1-q^2}{y_\lambda} & 0 & 0 & 0 & 0 & 0 \\
0\backslash/0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0\backslash/10 & 0 & 0 & 0 & 0 & 0 & \frac{1-q^2}{y_\lambda} & 0 & \frac{qx_\lambda}{y_\lambda} & 0 \\
10\backslash/1 & 0 & 0 & \frac{qx_\lambda}{y_\lambda} & 0 & 0 & 0 & \frac{(1-q^2)(1-\beta x_\lambda)}{y_\lambda} & 0 & 0 \\
10\backslash/0 & 0 & 0 & 0 & 0 & 0 & \frac{Q(\beta)qx_\lambda}{y_\lambda} & 0 & \frac{(1-q^2)(1-\beta x_\lambda)}{y_\lambda} & 0 \\
10\backslash/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

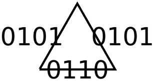
Proof of puzzle rule: puzzle R-matrix

$R_{gr}(\beta, x_\lambda) =$ 

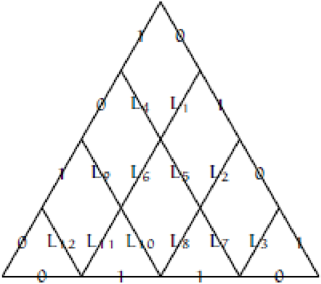
$=$

$$
\begin{array}{c|ccccccccc}
 & 1/\backslash 1 & 1/\backslash 0 & 1/\backslash 10 & 0/\backslash 1 & 0/\backslash 0 & 0/\backslash 10 & 10/\backslash 1 & 10/\backslash 0 & 10/\backslash 10 \\
\hline
1\backslash/1 & 1 & 0 & 0 & 0 & 0 & \frac{(1-q^2)(1-\beta x_\lambda)}{y_\lambda} & 0 & 0 & 0 \\
1\backslash/0 & 0 & 0 & 0 & \frac{q x_\lambda}{y_\lambda} & 0 & 0 & 0 & 0 & 0 \\
1\backslash/10 & 0 & 0 & 0 & 0 & \frac{1-q^2}{y_\lambda} & 0 & 1 & 0 & 0 \\
0\backslash/1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q(\beta)(q^2-1)(1-\beta x_\lambda)}{q y_\lambda} \\
0\backslash/0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{(1-q^2)(1-\beta x_\lambda)}{y_\lambda} & 0 & 0 \\
0\backslash/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Q(\beta) q x_\lambda}{y_\lambda} & 0 \\
10\backslash/1 & 0 & 0 & \frac{Q(\beta) q x_\lambda}{y_\lambda} & 0 & 0 & 0 & 0 & 0 & 0 \\
10\backslash/0 & \frac{1-q^2}{y_\lambda} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
10\backslash/10 & 0 & \frac{q(q^2-1)}{y_\lambda} & 0 & 0 & 0 & 0 & 0 & 0 & Q(\beta)
\end{array}
$$

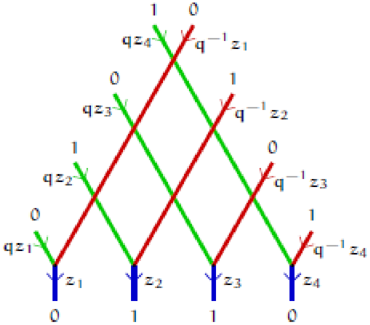
Proof of the puzzle rule

The following diagram equals .

$$\sum_{L_1,\ldots,L_{12}\in\{0,1,10\}}$$

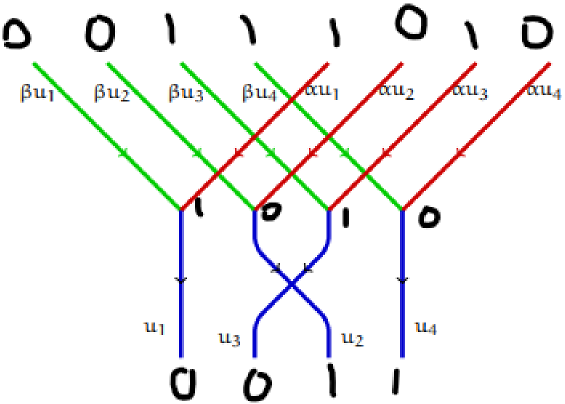


$=$



Proof of the puzzle rule

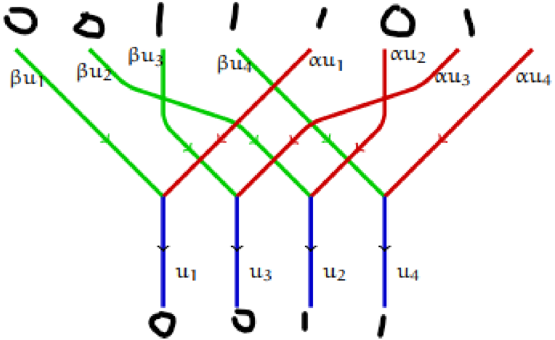
The following diagram computes ${}^{0011}\triangle^{1010}_{1010}\; S_{1010}|_{0101}$.



Removing 1010 in the center, it computes $\sum_\nu {}^{0011}\triangle^{1010}_{\nu}\; S_\nu|_{0101}$.

Proof of the puzzle rule

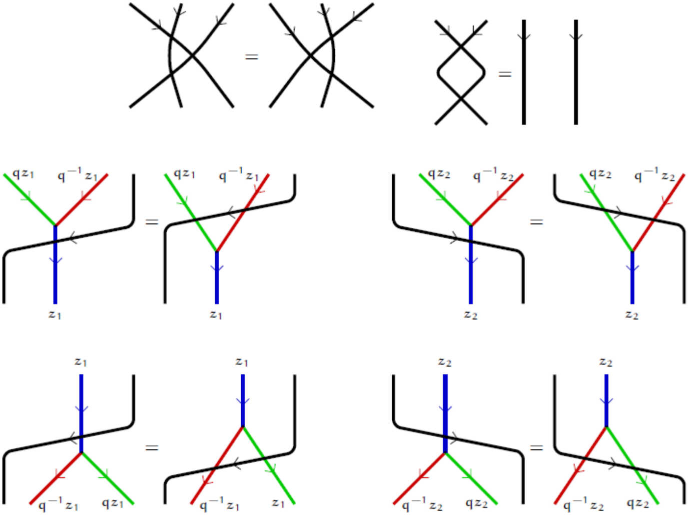
The following diagram computes $S_{0011}|_{0101} \cdot S_{1010}|_{0101}$ (I am sweeping details under the rug!) Note: red and green matrices “equal" blue matrix (almost).



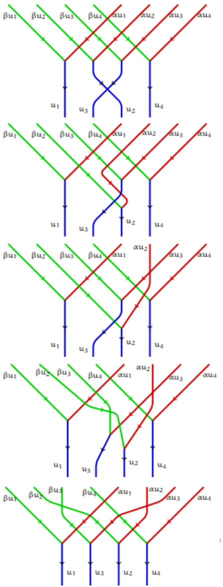
Must prove that this diagram equals previous one! Equality of formula at all restrictions implies equality of classes.

Proof of the puzzle rule

The following hold!



Proof of the puzzle rule



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