Finite Reflection Groups

Raj Gandhi
Supervised by: K.Zaynullin

University of Ottawa

September 8, 2016

Reflection Groups

Definition

The **Reflection** of a vector t with respect to a fixed vector a in a real euclidean space is defined by

$$s_a t = t - \frac{2(t,a)}{(a,a)} a$$

A **Reflection Group**, W, is a group that is generated by a set of such linear operators s_a

Example

A **Dihedral Group** is a group that is generated by the rotations and reflections a two-dimensional polygon that result in new orientations of the polygon
The rotations and reflections of an m sided polygon can be achieved by reflections of the polygon over its diagonals
Thus any **Dihedral Group** of order 2m can be thought of as a **Reflection Group**

Reflection Groups

Definition

The **Reflection** of a vector t with respect to a fixed vector a in a real euclidean space is defined by

$$s_a t = t - \frac{2(t,a)}{(a,a)} a$$

A **Reflection Group**, W, is a group that is generated by a set of such linear operators s_a

Example

A **Dihedral Group** is a group that is generated by the rotations and reflections a two-dimensional polygon that result in new orientations of the polygon

The rotations and reflections of an m sided polygon can be achieved by reflections of the polygon over its diagonals

Thus any **Dihedral Group** of order 2m can be thought of as a **Reflection Group**

Reflection Groups

Definition

The **Reflection** of a vector t with respect to a fixed vector a in a real euclidean space is defined by

$$s_a t = t - \frac{2(t,a)}{(a,a)} a$$

A **Reflection Group**, W, is a group that is generated by a set of such linear operators s_a

Example

A **Dihedral Group** is a group that is generated by the rotations and reflections a two-dimensional polygon that result in new orientations of the polygon

The rotations and reflections of an m sided polygon can be achieved by reflections of the polygon over its diagonals

Thus any **Dihedral Group** of order 2m can be thought of as a **Reflection Group**

Reflection Groups

Definition

The **Reflection** of a vector t with respect to a fixed vector a in a real euclidean space is defined by

$$s_a t = t - \frac{2(t,a)}{(a,a)} a$$

A **Reflection Group**, W, is a group that is generated by a set of such linear operators s_a

Example

A **Dihedral Group** is a group that is generated by the rotations and reflections a two-dimensional polygon that result in new orientations of the polygon

The rotations and reflections of an m sided polygon can be achieved by reflections of the polygon over its diagonals

Thus any **Dihedral Group** of order 2m can be thought of as a **Reflection Group**

Reflection Groups

Definition

The **Reflection** of a vector t with respect to a fixed vector a in a real euclidean space is defined by

$$s_a t = t - \frac{2(t,a)}{(a,a)} a$$

A **Reflection Group**, W, is a group that is generated by a set of such linear operators s_a

Example

A **Dihedral Group** is a group that is generated by the rotations and reflections a two-dimensional polygon that result in new orientations of the polygon

The rotations and reflections of an m sided polygon can be achieved by reflections of the polygon over its diagonals

Thus any **Dihedral Group** of order 2m can be thought of as a **Reflection Group**

Reflection Groups

Definition

The **Reflection** of a vector t with respect to a fixed vector a in a real euclidean space is defined by

$$s_a t = t - \frac{2(t,a)}{(a,a)} a$$

A **Reflection Group**, W, is a group that is generated by a set of such linear operators s_a

Example

A **Dihedral Group** is a group that is generated by the rotations and reflections a two-dimensional polygon that result in new orientations of the polygon

The rotations and reflections of an m sided polygon can be achieved by reflections of the polygon over its diagonals

Thus any **Dihedral Group** of order 2m can be thought of as a **Reflection Group**

Reflection Groups

Example

Any **Symmetric Group** can be thought of as a subgroup of the group of orthogonal matrices

Transposing two basis vectors of an orthogonal matrix is a **Reflection** which sends some vector $e_i - e_j$ to its negative while fixing pointwise every other vector of the matrix

Every **Symmetric Group** is also generated by such transpositions

Therefore, every **Symmetric Group** can also be realized as a **Reflection Group**

Reflection Groups

Example

Any **Symmetric Group** can be thought of as a subgroup of the group of orthogonal matrices

Transposing two basis vectors of an orthogonal matrix is a **Reflection** which sends some vector $e_i - e_j$ to its negative while fixing pointwise every other vector of the matrix

Every **Symmetric Group** is also generated by such transpositions

Therefore, every **Symmetric Group** can also be realized as a **Reflection Group**

Reflection Groups

Example

Any **Symmetric Group** can be thought of as a subgroup of the group of orthogonal matrices

Transposing two basis vectors of an orthogonal matrix is a **Reflection** which sends some vector $e_i - e_j$ to its negative while fixing pointwise every other vector of the matrix

Every **Symmetric Group** is also generated by such transpositions

Therefore, every **Symmetric Group** can also be realized as a **Reflection Group**

Reflection Groups

Example

Any **Symmetric Group** can be thought of as a subgroup of the group of orthogonal matrices

Transposing two basis vectors of an orthogonal matrix is a **Reflection** which sends some vector $e_i - e_j$ to its negative while fixing pointwise every other vector of the matrix

Every **Symmetric Group** is also generated by such transpositions

Therefore, every **Symmetric Group** can also be realized as a **Reflection Group**

Reflection Groups

Example

Any **Symmetric Group** can be thought of as a subgroup of the group of orthogonal matrices

Transposing two basis vectors of an orthogonal matrix is a **Reflection** which sends some vector $e_i - e_j$ to its negative while fixing pointwise every other vector of the matrix

Every **Symmetric Group** is also generated by such transpositions

Therefore, every **Symmetric Group** can also be realized as a **Reflection Group**

Root Systems

A root system R is a set of vectors that obeys the following axioms:

1. $R \cap \mathbf{c}a = \{a, -a\} \qquad \forall a \in R$
2. $s_a R = R \qquad \forall a \in R$

Example

The Dihedral Group of order 4 preserves these eight vectors:

$$\pm(1,0), \pm(1,1), \pm(0,1), \pm(-1,1)$$

If we think of this Dihedral Group as a Reflection Group, these vectors form a root system with associated reflection group W of generators associated with each vector

Root Systems

A root system R is a set of vectors that obeys the following axioms:

1. $$R \cap \mathbf{c}a = \{a, -a\} \qquad \forall a \in R$$
2. $$s_a R = R \qquad \forall a \in R$$

Example

The Dihedral Group of order 4 preserves these eight vectors:

$$\pm(1,0), \pm(1,1), \pm(0,1), \pm(-1,1)$$

If we think of this Dihedral Group as a Reflection Group, these vectors form a root system with associated reflection group W of generators associated with each vector

Root Systems

A root system R is a set of vectors that obeys the following axioms:

1. $$R \cap \mathbf{c}a = \{a, -a\} \qquad \forall a \in R$$
2. $$s_a R = R \qquad \forall a \in R$$

Example

The Dihedral Group of order 4 preserves these eight vectors:

$$\pm(1,0), \pm(1,1), \pm(0,1), \pm(-1,1)$$

If we think of this Dihedral Group as a Reflection Group, these vectors form a root system with associated reflection group W of generators associated with each vector

A root system R is a set of vectors that obeys the following axioms:

1. $$R \cap \mathbf{c}a = \{a, -a\} \qquad \forall a \in R$$
2. $$s_a R = R \qquad \forall a \in R$$

Example

The Dihedral Group of order 4 preserves these eight vectors:

$$\pm(1,0), \pm(1,1), \pm(0,1), \pm(-1,1)$$

If we think of this Dihedral Group as a Reflection Group, these vectors form a root system with associated reflection group W of generators associated with each vector

Root Systems

A root system R is a set of vectors that obeys the following axioms:

1. $$R \cap \mathbf{c}a = \{a, -a\} \qquad \forall a \in R$$
2. $$s_a R = R \qquad \forall a \in R$$

Example

The Dihedral Group of order 4 preserves these eight vectors:

$$\pm(1,0), \pm(1,1), \pm(0,1), \pm(-1,1)$$

If we think of this Dihedral Group as a Reflection Group, these vectors form a root system with associated reflection group W of generators associated with each vector

Root Systems

A root system R is a set of vectors that obeys the following axioms:

1. $$R \cap \mathbf{c}a = \{a, -a\} \qquad \forall a \in R$$
2. $$s_a R = R \qquad \forall a \in R$$

Example

The Dihedral Group of order 4 preserves these eight vectors:

$$\pm(1,0), \pm(1,1), \pm(0,1), \pm(-1,1)$$

If we think of this Dihedral Group as a Reflection Group, these vectors form a root system with associated reflection group W of generators associated with each vector

Positive and Simple Systems

Definition

A **Positive System** Π is a partition of the Root System obtained from a linear combination of an ordered basis of V with strictly positive coefficients
A **Negative System** $-\Pi$ is a partition of the Root System obtained from a linear combination of an ordered basis of V with with strictly negative coefficients

$$R = \Pi \cup -\Pi$$

A **Simple System** is a vector space basis for the roots in R
Every root, B, in the Root System can be obtained from some linear combination of simple roots with coefficients, a_i, all of the same sign

$$B = \sum_i c_{a_i} a_i$$

Positive and Simple Systems

Definition

A **Positive System** Π is a partition of the Root System obtained from a linear combination of an ordered basis of V with strictly positive coefficients

A **Negative System** $-\Pi$ is a partition of the Root System obtained from a linear combination of an ordered basis of V with with strictly negative coefficients

$$R = \Pi \cup -\Pi$$

A **Simple System** is a vector space basis for the roots in R

Every root, B, in the Root System can be obtained from some linear combination of simple roots with coefficients, a_i, all of the same sign

$$B = \sum_i c_{a_i} a_i$$

Positive and Simple Systems

Definition

A **Positive System** Π is a partition of the Root System obtained from a linear combination of an ordered basis of V with strictly positive coefficients
A **Negative System** $-\Pi$ is a partition of the Root System obtained from a linear combination of an ordered basis of V with with strictly negative coefficients

$$R = \Pi \cup -\Pi$$

A **Simple System** is a vector space basis for the roots in R
Every root, B, in the Root System can be obtained from some linear combination of simple roots with coefficients, a_i, all of the same sign

$$B = \sum_i c_{a_i} a_i$$

Positive and Simple Systems

Definition

A **Positive System** Π is a partition of the Root System obtained from a linear combination of an ordered basis of V with strictly positive coefficients
A **Negative System** $-\Pi$ is a partition of the Root System obtained from a linear combination of an ordered basis of V with with strictly negative coefficients

$$R = \Pi \cup -\Pi$$

A **Simple System** is a vector space basis for the roots in R
Every root, B, in the Root System can be obtained from some linear combination of simple roots with coefficients, a_i, all of the same sign

$$B = \sum_i c_{a_i} a_i$$

Positive and Simple Systems

Definition

A **Positive System** Π is a partition of the Root System obtained from a linear combination of an ordered basis of V with strictly positive coefficients
A **Negative System** $-\Pi$ is a partition of the Root System obtained from a linear combination of an ordered basis of V with with strictly negative coefficients

$$R = \Pi \cup -\Pi$$

A **Simple System** is a vector space basis for the roots in R
Every root, B, in the Root System can be obtained from some linear combination of simple roots with coefficients, a_i, all of the same sign

$$B = \sum_i c_{a_i} a_i$$

Positive and Simple Systems

How do we know Simple Systems exist?

Proof.

Take a positive system Π in R

Choose a set of roots in Π that are not expressible as a linear combination of the other roots in Π with strictly positive coefficients

This is a simple system in Π, which implies that simple systems exist

□

Example

A simple system for the Symmetric Group is the set

$$S = \{e_i - e_j | i = j + 1, 0 < j < n\}$$

Positive and Simple Systems

How do we know Simple Systems exist?

Proof.

Take a positive system Π in R

Choose a set of roots in Π that are not expressible as a linear combination of the other roots in Π with strictly positive coefficients

This is a simple system in Π, which implies that simple systems exist

□

Example

A simple system for the Symmetric Group is the set

$$S = \{e_i - e_j | i = j + 1, 0 < j < n\}$$

Positive and Simple Systems

How do we know Simple Systems exist?

Proof.

Take a positive system Π in R

Choose a set of roots in Π that are not expressible as a linear combination of the other roots in Π with strictly positive coefficients

This is a simple system in Π, which implies that simple systems exist

□

Example

A simple system for the Symmetric Group is the set

$$S = \{e_i - e_j | i = j + 1, 0 < j < n\}$$

Positive and Simple Systems

How do we know Simple Systems exist?

Proof.

Take a positive system Π in R

Choose a set of roots in Π that are not expressible as a linear combination of the other roots in Π with strictly positive coefficients

This is a simple system in Π, which implies that simple systems exist

□

Example

A simple system for the Symmetric Group is the set

$$S = \{e_i - e_j | i = j + 1, 0 < j < n\}$$

Positive and Simple Systems

How do we know Simple Systems exist?

Proof.

Take a positive system Π in R

Choose a set of roots in Π that are not expressible as a linear combination of the other roots in Π with strictly positive coefficients

This is a simple system in Π, which implies that simple systems exist

□

Example

A simple system for the Symmetric Group is the set

$$S = \{e_i - e_j | i = j + 1, 0 < j < n\}$$

Positive and Simple Systems

Theorem

$$(a, b) \leq 0 \qquad \forall a, b \in \triangle$$

Theorem

Let $\triangle$ be a simple system contained in Π. If $a \in \triangle$, then $s_a(\Pi/\{a\}) = \Pi/\{a\}$

Proof.

$\exists B \in \Pi$ *that can be written as a linear combination of* $\triangle$ *with strictly positive coefficients*

But $s_a B = B - \frac{2(B,a)}{(a,a)} a = B - \frac{2\sum_k c_k(k,a)}{(a,a)} a > 0, B \neq a, k \in \triangle$

If $B = a$*, then* $s_a B = s_a a = -a$

Thus the only positive root made negative by s_a *is* a

Positive and Simple Systems

Theorem

$$(a, b) \leq 0 \qquad \forall a, b \in \triangle$$

Theorem

Let $\triangle$ be a simple system contained in Π. If $a \in \triangle$, then $s_a(\Pi/\{a\}) = \Pi/\{a\}$

Proof.

$\exists B \in \Pi$ *that can be written as a linear combination of* $\triangle$ *with strictly positive coefficients*

But $s_a B = B - \frac{2(B,a)}{(a,a)} a = B - \frac{2\sum_k c_k(k,a)}{(a,a)} a > 0, B \neq a, k \in \triangle$

If $B = a$, *then* $s_a B = s_a a = -a$

Thus the only positive root made negative by s_a *is* a

Positive and Simple Systems

Theorem

$$(a, b) \leq 0 \qquad \forall a, b \in \triangle$$

Theorem

Let $\triangle$ be a simple system contained in Π. If $a \in \triangle$, then $s_a(\Pi/\{a\}) = \Pi/\{a\}$

Proof.

$\exists B \in \Pi$ *that can be written as a linear combination of* $\triangle$ *with strictly positive coefficients*

But $s_a B = B - \frac{2(B,a)}{(a,a)} a = B - \frac{2\sum_k c_k(k,a)}{(a,a)} a > 0, B \neq a, k \in \triangle$

If $B = a$, *then* $s_a B = s_a a = -a$

Thus the only positive root made negative by s_a *is* a

Positive and Simple Systems

Theorem

$$(a, b) \leq 0 \qquad \forall a, b \in \triangle$$

Theorem

Let $\triangle$ be a simple system contained in Π. If $a \in \triangle$, then $s_a(\Pi/\{a\}) = \Pi/\{a\}$

Proof.

$\exists B \in \Pi$ that can be written as a linear combination of $\triangle$ with strictly positive coefficients

But $s_a B = B - \frac{2(B,a)}{(a,a)} a = B - \frac{2\sum_k c_k(k,a)}{(a,a)} a > 0, B \neq a, k \in \triangle$

If $B = a$, then $s_a B = s_a a = -a$

Thus the only positive root made negative by s_a is a

Positive and Simple Systems

Theorem

$$(a, b) \leq 0 \qquad \forall a, b \in \triangle$$

Theorem

Let $\triangle$ be a simple system contained in Π. If $a \in \triangle$, then $s_a(\Pi/\{a\}) = \Pi/\{a\}$

Proof.

$\exists B \in \Pi$ *that can be written as a linear combination of* $\triangle$ *with strictly positive coefficients*

But $s_a B = B - \frac{2(B,a)}{(a,a)} a = B - \frac{2\sum_k c_k(k,a)}{(a,a)} a > 0, B \neq a, k \in \triangle$

If $B = a$, *then* $s_a B = s_a a = -a$

Thus the only positive root made negative by s_a *is* a

Positive and Simple Systems

Theorem

$$(a, b) \leq 0 \qquad \forall a, b \in \triangle$$

Theorem

Let $\triangle$ be a simple system contained in Π. If $a \in \triangle$, then $s_a(\Pi/\{a\}) = \Pi/\{a\}$

Proof.

$\exists B \in \Pi$ that can be written as a linear combination of $\triangle$ with strictly positive coefficients

But $s_a B = B - \frac{2(B,a)}{(a,a)} a = B - \frac{2\sum_k c_k(k,a)}{(a,a)} a > 0, B \neq a, k \in \triangle$

If $B = a$, *then* $s_a B = s_a a = -a$

Thus the only positive root made negative by s_a is a

Positive and Simple Systems

Theorem

$$(a, b) \leq 0 \qquad \forall a, b \in \triangle$$

Theorem

Let $\triangle$ be a simple system contained in Π. If $a \in \triangle$, then $s_a(\Pi/\{a\}) = \Pi/\{a\}$

Proof.

$\exists B \in \Pi$ that can be written as a linear combination of $\triangle$ with strictly positive coefficients

But $s_a B = B - \frac{2(B,a)}{(a,a)} a = B - \frac{2\sum_k c_k(k,a)}{(a,a)} a > 0, B \neq a, k \in \triangle$

If $B = a$, *then* $s_a B = s_a a = -a$

Thus the only positive root made negative by s_a is a

Positive and Simple Systems

Theorem

Any two positive systems in R are conjugate under W

Proof.

Take two positive systems, Π and Π', in R

If $r = Card\{\Pi \cap -\Pi'\}$ and $r=0$, then $\Pi=-\Pi'$

If $r > 0$, then $\exists a \in \Pi'$ such that $a \in -\Pi$.
Thus take $a \in \Pi'$ and $a \notin \Pi$ so that $Card\{\Pi \cap s_a(-\Pi')\} = r\text{-}1$

Continuing this way we furnish an element w such that
$Card\{\Pi \cap w(-\Pi')\} = 0$ □

Positive and Simple Systems

Theorem

Any two positive systems in R are conjugate under W

Proof.

Take two positive systems, Π and Π', in R

If $r = Card\{\Pi \cap -\Pi'\}$ and r=0, then Π=$-\Pi'$

If $r > 0$, then $\exists a \in \Pi'$ such that $a \in -\Pi$.
Thus take $a \in \Pi'$ and $a \notin \Pi$ so that $Card\{\Pi \cap s_a(-\Pi')\} = $ r-1

Continuing this way we furnish an element w such that $Card\{\Pi \cap w(-\Pi')\} = 0$

□

Positive and Simple Systems

Theorem

Any two positive systems in R are conjugate under W

Proof.

Take two positive systems, Π *and* Π'*, in R*

If $r = Card\{\Pi \cap -\Pi'\}$ *and* $r=0$*, then* $\Pi = -\Pi'$

If $r > 0$*, then* $\exists a \in \Pi'$ *such that* $a \in -\Pi$.
Thus take $a \in \Pi'$ *and* $a \notin \Pi$ *so that* $Card\{\Pi \cap s_a(-\Pi')\} =$ *r-1*

Continuing this way we furnish an element w such that
$Card\{\Pi \cap w(-\Pi')\} = 0$ □

Positive and Simple Systems

Theorem

Any two positive systems in R are conjugate under W

Proof.

Take two positive systems, Π and Π', in R

If $r = Card\{\Pi \cap -\Pi'\}$ and r=0, then $\Pi = -\Pi'$

If $r > 0$, then $\exists a \in \Pi'$ such that $a \in -\Pi$.
Thus take $a \in \Pi'$ and $a \notin \Pi$ so that $Card\{\Pi \cap s_a(-\Pi')\} = r\text{-}1$

Continuing this way we furnish an element w such that $Card\{\Pi \cap w(-\Pi')\} = 0$ □

Positive and Simple Systems

Theorem

Any two positive systems in R are conjugate under W

Proof.

Take two positive systems, Π *and* Π'*, in R*

If $r = \mathit{Card}\{\Pi \cap -\Pi'\}$ *and r=0, then* $\Pi = -\Pi'$

If $r > 0$*, then* $\exists a \in \Pi'$ *such that* $a \in -\Pi$.
Thus take $a \in \Pi'$ *and* $a \notin \Pi$ *so that* $\mathit{Card}\{\Pi \cap s_a(-\Pi')\} = r\text{-}1$

Continuing this way we furnish an element w such that $\mathit{Card}\{\Pi \cap w(-\Pi')\} = 0$ □

Positive and Simple Systems

Theorem

Any two positive systems in R are conjugate under W

Proof.

Take two positive systems, Π and Π', in R

If $r = Card\{\Pi \cap -\Pi'\}$ *and r=0, then* $\Pi = -\Pi'$

If $r > 0$, *then* $\exists a \in \Pi'$ *such that* $a \in -\Pi$.
Thus take $a \in \Pi'$ *and* $a \notin \Pi$ *so that* $Card\{\Pi \cap s_a(-\Pi')\} = r\text{-}1$

Continuing this way we furnish an element w such that
$Card\{\Pi \cap w(-\Pi')\} = 0$ □

Properties of Reflection Groups

Theorem

For a fixed simple system $\triangle$, W is generated by simple reflections, $s_a (a \in \triangle)$

Theorem

Given $\triangle$, $\forall\ B \in R\ \exists w \in W$ such that $wB \in \triangle$

Definition

Take $w \in W$, where $w = s_{i_1} s_{i_2} \ldots s_{i_r}$
The **length** of w, defined by **Length Function** $l(w)$, is the smallest r for which w exists
Some properties of reflection combinations, ie: ww', useful for later proofs are

1. $l(ww') \leq l(w) + l(w')$ since max(l(ww'))=r+r'
2. $l(s_a w) = l(w) \pm 1$

Properties of Reflection Groups

Theorem

For a fixed simple system $\triangle$, W is generated by simple reflections, $s_a (a \in \triangle)$

Theorem

Given $\triangle$, $\forall\ B \in R\ \exists w \in W$ such that $wB \in \triangle$

Definition

Take $w \in W$, where $w = s_{i_1} s_{i_2} ... s_{i_r}$

The **length** of w, defined by **Length Function** $l(w)$, is the smallest r for which w exists

Some properties of reflection combinations, ie: ww', useful for later proofs are

1. $l(ww') \leq l(w) + l(w')$ since max(l(ww'))=r+r'
2. $l(s_a w) = l(w) \pm 1$

Properties of Reflection Groups

Theorem

For a fixed simple system $\triangle$, W is generated by simple reflections, $s_a (a \in \triangle)$

Theorem

Given $\triangle$, $\forall\ B \in R\ \exists w \in W$ such that $wB \in \triangle$

Definition

Take $w \in W$, where $w = s_{i_1} s_{i_2} \ldots s_{i_r}$

The **length** of w, defined by **Length Function** $l(w)$, is the smallest r for which w exists

Some properties of reflection combinations, ie: ww', useful for later proofs are

1. $l(ww') \leq l(w) + l(w')$ since max(l(ww'))=r+r'
2. $l(s_a w) = l(w) \pm 1$

Properties of Reflection Groups

Theorem

For a fixed simple system $\triangle$, W is generated by simple reflections, $s_a (a \in \triangle)$

Theorem

Given $\triangle$, $\forall\ B \in R\ \exists w \in W$ such that $wB \in \triangle$

Definition

Take $w \in W$, where $w = s_{i_1} s_{i_2} ... s_{i_r}$

The **length** of w, defined by **Length Function** $l(w)$, is the smallest r for which w exists

Some properties of reflection combinations, ie: ww', useful for later proofs are

1. $l(ww') \leq l(w) + l(w')$ since max(l(ww'))=r+r'
2. $l(s_a w) = l(w) \pm 1$

Properties of Reflection Groups

Theorem

For a fixed simple system $\triangle$, W is generated by simple reflections, $s_a (a \in \triangle)$

Theorem

Given $\triangle$, $\forall\ B \in R\ \exists w \in W$ such that $wB \in \triangle$

Definition

Take $w \in W$, where $w = s_{i_1} s_{i_2} ... s_{i_r}$
The **length** of w, defined by **Length Function** $l(w)$, is the smallest r for which w exists

Some properties of reflection combinations, ie: ww', useful for later proofs are

1. $l(ww') \leq l(w) + l(w')$ since max(l(ww'))=r+r'
2. $l(s_a w) = l(w) \pm 1$

Properties of Reflection Groups

Theorem

For a fixed simple system $\triangle$, W is generated by simple reflections, $s_a (a \in \triangle)$

Theorem

Given $\triangle$, $\forall\ B \in R\ \exists w \in W$ such that $wB \in \triangle$

Definition

Take $w \in W$, where $w = s_{i_1} s_{i_2} ... s_{i_r}$
The **length** of w, defined by **Length Function** $l(w)$, is the smallest r for which w exists
Some properties of reflection combinations, ie: ww', useful for later proofs are

1. $l(ww') \leq l(w) + l(w')$ since max(l(ww'))=r+r'
2. $l(s_a w) = l(w) \pm 1$

Properties of Reflection Groups

Theorem

For a fixed simple system $\triangle$, W is generated by simple reflections, $s_a (a \in \triangle)$

Theorem

Given $\triangle$, $\forall\ B \in R\ \exists w \in W$ such that $wB \in \triangle$

Definition

Take $w \in W$, where $w = s_{i_1} s_{i_2} ... s_{i_r}$

The **length** of w, defined by **Length Function** $l(w)$, is the smallest r for which w exists

Some properties of reflection combinations, ie: ww', useful for later proofs are

1. $l(ww') \leq l(w) + l(w')$ since max(l(ww'))=r+r'
2. $l(s_a w) = l(w) \pm 1$

Properties of Reflection Groups

Theorem

For a fixed simple system $\triangle$, W is generated by simple reflections, $s_a (a \in \triangle)$

Theorem

Given $\triangle$, $\forall\ B \in R\ \exists w \in W$ such that $wB \in \triangle$

Definition

Take $w \in W$, where $w = s_{i_1} s_{i_2} ... s_{i_r}$
The **length** of w, defined by **Length Function** $l(w)$, is the smallest r for which w exists
Some properties of reflection combinations, ie: ww', useful for later proofs are

1. $l(ww') \leq l(w) + l(w')$ since max(l(ww'))=r+r'
2. $l(s_a w) = l(w) \pm 1$

Application of the Length Function

Given W with an associated root system, the number of positive roots made negative by w can be characterized by the equation:

$$n(w) = Card\{\Pi \cap w^{-1}(-\Pi)\}$$

From this definition and the properties of the length function, we can prove that

1. $$wa > 0 \implies n(ws_a) = n(w) + 1$$
2. $$wa < 0 \implies n(ws_a) = n(w) - 1$$

Corollary: Since n(w) can increase by at most 1 for each added generator, $n(w) \leq r$

Application of the Length Function

Given W with an associated root system, the number of positive roots made negative by w can be characterized by the equation:

$$n(w) = Card\{\Pi \cap w^{-1}(-\Pi)\}$$

From this definition and the properties of the length function, we can prove that

1. $$wa > 0 \Longrightarrow n(ws_a) = n(w) + 1$$
2. $$wa < 0 \Longrightarrow n(ws_a) = n(w) - 1$$

Corollary: Since n(w) can increase by at most 1 for each added generator, $n(w) \leq r$

Application of the Length Function

Given W with an associated root system, the number of positive roots made negative by w can be characterized by the equation:

$$n(w) = Card\{\Pi \cap w^{-1}(-\Pi)\}$$

From this definition and the properties of the length function, we can prove that

1. $$wa > 0 \Longrightarrow n(ws_a) = n(w) + 1$$
2. $$wa < 0 \Longrightarrow n(ws_a) = n(w) - 1$$

Corollary: Since n(w) can increase by at most 1 for each added generator, $n(w) \leq r$

Application of the Length Function

Given W with an associated root system, the number of positive roots made negative by w can be characterized by the equation:

$$n(w) = Card\{\Pi \cap w^{-1}(-\Pi)\}$$

From this definition and the properties of the length function, we can prove that

1. $$wa > 0 \implies n(ws_a) = n(w) + 1$$
2. $$wa < 0 \implies n(ws_a) = n(w) - 1$$

Corollary: Since n(w) can increase by at most 1 for each added generator, $n(w) \leq r$

Application of the Length Function

Given W with an associated root system, the number of positive roots made negative by w can be characterized by the equation:

$$n(w) = Card\{\Pi \cap w^{-1}(-\Pi)\}$$

From this definition and the properties of the length function, we can prove that

1. $$wa > 0 \Longrightarrow n(ws_a) = n(w) + 1$$
2. $$wa < 0 \Longrightarrow n(ws_a) = n(w) - 1$$

Corollary: Since n(w) can increase by at most 1 for each added generator, $n(w) \leq r$

Properties of Reflection Groups

Theorem

The Deletion Condition

Fix a simple system $\triangle$. Take $w = s_1 ... s_r$ with $w \in W$ as a product of simple reflections. Suppose $n(w) < r$. Then there are indices $1 \leq i < j \leq r$ such that

1. $$a_i = (s_{i+1} ... s_{j-1}) a_j$$
2. $$s_{i+1} \ldots s_j = s_i \ldots s_{j-1}$$
3. $$w = s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_r$$

Proof.

$$w = s_1 ... s_i s_{i+1} ... s_{j-1} s_j s_{j+1} ... s_r = s_1 ... s_i (s_i ... s_{j-1}) s_{j+1} ... s_r = s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_r$$

□

Properties of Reflection Groups

Theorem

The Deletion Condition

Fix a simple system $\triangle$. Take $w = s_1 \ldots s_r$ with $w \in W$ as a product of simple reflections. Suppose $n(w) < r$. Then there are indices $1 \leq i < j \leq r$ such that

1. $$a_i = (s_{i+1} \ldots s_{j-1}) a_j$$
2. $$s_{i+1} \ldots s_j = s_i \ldots s_{j-1}$$
3. $$w = s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_r$$

Proof.

$$w = s_1 \ldots s_i s_{i+1} \ldots s_{j-1} s_j s_{j+1} \ldots s_r = s_1 \ldots s_i (s_i \ldots s_{j-1}) s_{j+1} \ldots s_r =$$
$$s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_r$$
□

Properties of Reflection Groups

Theorem

The Deletion Condition

Fix a simple system $\triangle$. Take $w = s_1 ... s_r$ with $w \in W$ as a product of simple reflections. Suppose $n(w) < r$. Then there are indices $1 \leq i < j \leq r$ such that

1. $$a_i = (s_{i+1} ... s_{j-1}) a_j$$
2. $$s_{i+1} ... s_j = s_i ... s_{j-1}$$
3. $$w = s_1 ... s_i ... s_j ... s_r$$

Proof.

$$w = s_1 ... s_i s_{i+1} ... s_{j-1} s_j s_{j+1} ... s_r = s_1 ... s_i (s_i ... s_{j-1}) s_{j+1} ... s_r = s_1 ... s_i ... s_j ... s_r$$

□

Properties of Reflection Groups

Theorem

The Deletion Condition

Fix a simple system $\triangle$. Take $w = s_1 ... s_r$ with $w \in W$ as a product of simple reflections. Suppose $n(w) < r$. Then there are indices $1 \leq i < j \leq r$ such that

1. $$a_i = (s_{i+1} ... s_{j-1}) a_j$$
2. $$s_{i+1} \dots s_j = s_i \dots s_{j-1}$$
3. $$w = s_1 \dots s_i \dots s_j \dots s_r$$

Proof.

$$w = s_1 \dots s_i s_{i+1} \dots s_{j-1} s_j s_{j+1} \dots s_r = s_1 \dots s_i (s_i \dots s_{j-1}) s_{j+1} \dots s_r =$$
$$s_1 \dots s_i \dots s_j \dots s_r$$

□

Properties of Reflection Groups

Theorem

The Deletion Condition

Fix a simple system $\triangle$. Take $w = s_1 ... s_r$ with $w \in W$ as a product of simple reflections. Suppose $n(w) < r$. Then there are indices $1 \leq i < j \leq r$ such that

1. $$a_i = (s_{i+1} ... s_{j-1}) a_j$$
2. $$s_{i+1} ... s_j = s_i ... s_{j-1}$$
3. $$w = s_1 ... \hat{s}_i ... \hat{s}_j ... s_r$$

Proof.

$$w = s_1 ... s_i s_{i+1} ... s_{j-1} s_j s_{j+1} ... s_r = s_1 ... s_i (s_i ... s_{j-1}) s_{j+1} ... s_r = s_1 ... \hat{s}_i ... \hat{s}_j ... s_r$$

□

Properties of Reflection Groups

Theorem

The Deletion Condition

Fix a simple system $\triangle$. Take $w = s_1 ... s_r$ with $w \in W$ as a product of simple reflections. Suppose $n(w) < r$. Then there are indices $1 \leq i < j \leq r$ such that

1. $$a_i = (s_{i+1} ... s_{j-1}) a_j$$
2. $$s_{i+1} ... s_j = s_i ... s_{j-1}$$
3. $$w = s_1 ... s_i ... s_j ... s_r$$

Proof.

$w = s_1 ... s_i s_{i+1} ... s_{j-1} s_j s_{j+1} ... s_r = s_1 ... s_i (s_i ... s_{j-1}) s_{j+1} ... s_r =$

$$s_1 ... s_i ... s_j ... s_r$$

□

Properties of Reflection Groups

Theorem

The Deletion Condition

Fix a simple system $\triangle$. Take $w = s_1 ... s_r$ with $w \in W$ as a product of simple reflections. Suppose $n(w) < r$. Then there are indices $1 \leq i < j \leq r$ such that

1. $$a_i = (s_{i+1} ... s_{j-1}) a_j$$
2. $$s_{i+1} ... s_j = s_i ... s_{j-1}$$
3. $$w = s_1 ... \hat{s}_i ... \hat{s}_j ... s_r$$

Proof.

$$w = s_1 ... s_i s_{i+1} ... s_{j-1} s_j s_{j+1} ... s_r = s_1 ... s_i (s_i ... s_{j-1}) s_{j+1} ... s_r =$$
$$s_1 ... \hat{s}_i ... \hat{s}_j ... s_r$$

□

Properties of Reflection Groups

Theorem

If $w \in W$ is reduced, then $n(w) = l(w)$

Proof.

We already know $n(w) \leq l(w)$

If $n(w) < l(w) = r$, then by the **Deletion Condition**, *$l(w)$ is equal to a product of r-2 simple reflections*

Since $l(w) = r$, we have a contradiction, forcing $n(w) = l(w)$ □

Properties of Reflection Groups

Theorem

If $w \in W$ is reduced, then $n(w) = l(w)$

Proof.

We already know $n(w) \leq l(w)$

If $n(w) < l(w) = r$, then by the **Deletion Condition**, *$l(w)$ is equal to a product of r-2 simple reflections*

Since $l(w) = r$, we have a contradiction, forcing $n(w) = l(w)$ □

Properties of Reflection Groups

Theorem

If $w \in W$ is reduced, then $n(w) = l(w)$

Proof.

We already know $n(w) \leq l(w)$

If $n(w) < l(w) = r$, then by the **Deletion Condition**, *l(w) is equal to a product of r-2 simple reflections*

Since $l(w) = r$, we have a contradiction, forcing $n(w) = l(w)$ □

Theorem

If $w \in W$ is reduced, then $n(w) = l(w)$

Proof.

We already know $n(w) \leq l(w)$

If $n(w) < l(w) = r$, then by the **Deletion Condition**, *l(w) is equal to a product of r-2 simple reflections*

Since $l(w) = r$, we have a contradiction, forcing $n(w) = l(w)$ □

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Definition

Fix a simple system Δ in R with an associated reflection group W. Then a **Coxeter Group** S is a group that generates W and is subject only to the relations

$$S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$$

where $m(a, b)$ is the order of $s_a s_b$ in W

Big Question: Can every reflection group W be generated by a Coxeter Group?

Answer: Yes!

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Definition

Fix a simple system Δ in R with an associated reflection group W. Then a **Coxeter Group** S is a group that generates W and is subject only to the relations

$$S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$$

where $m(a,b)$ is the order of $s_a s_b$ in W

Big Question: Can every reflection group W be generated by a Coxeter Group?

Answer: Yes!

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Definition

Fix a simple system Δ in R with an associated reflection group W. Then a **Coxeter Group** S is a group that generates W and is subject only to the relations

$$S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$$

where $m(a, b)$ is the order of $s_a s_b$ in W

Big Question: Can every reflection group W be generated by a Coxeter Group?

Answer: Yes!

Definition

Fix a simple system Δ in R with an associated reflection group W. Then a **Coxeter Group** S is a group that generates W and is subject only to the relations

$$S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$$

where $m(a, b)$ is the order of $s_a s_b$ in W

Big Question: Can every reflection group W be generated by a Coxeter Group?

Answer: Yes!

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

Since the collection of relations $s_1 ... s_r = 1$ in W completely describes W, we will show by induction that
$(S \implies s_{i_1} ... s_{i_k} = 1, k < r) \implies (S \implies s_1 ... s_r = 1$ (1) for every relation of r reflections in W)
Note: $r = 2q$ for $q \in Z$
Note: The base case of q=1 holds since $s_1 s_2 = 1$ implies that $s_1 = s_2^{-1}$ so $s_1 = s_2$, and realize that $(s_i s_i)^1 = 1$
Now rewrite (1) as $s_1 ... s_{q+1} = s_r ... s_{q+2}$; since l(right side)=q-1, this means l(left side)=q+1
By the Deletion Condition, there are indices $1 \leq i < j \leq q+1$ such that

$$s_i ... s_{j+1} = s_{i+1} ... s_j \ (2) \iff$$
$$s_i ... s_{j+1} s_j ... s_{i+1} = 1 \ \ (3)$$

Since we assume (3) is implied by S if it has less than r reflections, take $l(3) < r$. Then the Deletion Condition says that we may omit two factors

Proof.

Since the collection of relations $s_1 ... s_r = 1$ in W completely describes W, we will show by induction that
$(S \implies s_{i_1} ... s_{i_k} = 1, k < r) \implies (S \implies s_1 ... s_r = 1$ (1) for every relation of r reflections in W)
Note: $r = 2q$ for $q \in Z$
Note: The base case of q=1 holds since $s_1 s_2 = 1$ implies that $s_1 = s_2^{-1}$ so $s_1 = s_2$, and realize that $(s_i s_i)^1 = 1$
Now rewrite (1) as $s_1 ... s_{q+1} = s_r ... s_{q+2}$; since l(right side)=q-1, this means l(left side)=q+1
By the Deletion Condition, there are indices $1 \leq i < j \leq q+1$ such that

$$s_i ... s_{j+1} = s_{i+1} ... s_j \text{ (2)} \iff$$
$$s_i ... s_{j+1} s_j ... s_{i+1} = 1 \text{ (3)}$$

Since we assume (3) is implied by S if it has less than r reflections, take $l(3) < r$. Then the Deletion Condition says that we may omit two factors

Proof.

Since the collection of relations $s_1 \ldots s_r = 1$ in W completely describes W, we will show by induction that
$(S \implies s_{i_1} \ldots s_{i_k} = 1, k < r) \implies (S \implies s_1 \ldots s_r = 1$ (1) for every relation of r reflections in W)
Note: $r = 2q$ for $q \in Z$
Note: The base case of q=1 holds since $s_1 s_2 = 1$ implies that $s_1 = s_2^{-1}$ so $s_1 = s_2$, and realize that $(s_i s_i)^1 = 1$
Now rewrite (1) as $s_1 \ldots s_{q+1} = s_r \ldots s_{q+2}$; since l(right side)=q-1, this means l(left side)=q+1
By the Deletion Condition, there are indices $1 \leq i < j \leq q+1$ such that

$$s_i \ldots s_{j+1} = s_{i+1} \ldots s_j \text{ (2)} \iff$$
$$s_i \ldots s_{j+1} s_j \ldots s_{i+1} = 1 \text{ (3)}$$

Since we assume (3) is implied by S if it has less than r reflections, take $l(3) < r$. Then the Deletion Condition says that we may omit two factors

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

Since the collection of relations $s_1 ... s_r = 1$ in W completely describes W, we will show by induction that

$(S \implies s_{i_1} ... s_{i_k} = 1, k < r) \implies (S \implies s_1 ... s_r = 1$ (1) for every relation of r reflections in W)

Note: $r = 2q$ for $q \in Z$

Note: The base case of q=1 holds since $s_1 s_2 = 1$ implies that $s_1 = s_2^{-1}$ so $s_1 = s_2$, and realize that $(s_i s_i)^1 = 1$

Now rewrite (1) as $s_1 ... s_{q+1} = s_r ... s_{q+2}$; since l(right side)=q-1, this means l(left side)=q+1

By the Deletion Condition, there are indices $1 \leq i < j \leq q+1$ such that

$$s_i ... s_{j+1} = s_{i+1} ... s_j \text{ (2)} \iff$$
$$s_i ... s_{j+1} s_j ... s_{i+1} = 1 \text{ (3)}$$

Since we assume (3) is implied by S if it has less than r reflections, take $l(3) < r$. Then the Deletion Condition says that we may omit two factors

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

Since the collection of relations $s_1 ... s_r = 1$ in W completely describes W, we will show by induction that
$(S \Longrightarrow s_{i_1} ... s_{i_k} = 1, k < r) \Longrightarrow (S \Longrightarrow s_1 ... s_r = 1$ (1) for every relation of r reflections in W)
Note: $r = 2q$ for $q \in Z$
Note: The base case of q=1 holds since $s_1 s_2 = 1$ implies that $s_1 = s_2^{-1}$ so $s_1 = s_2$, and realize that $(s_i s_i)^1 = 1$
Now rewrite (1) as $s_1 ... s_{q+1} = s_r ... s_{q+2}$; since l(right side)=q-1, this means l(left side)=q+1
By the Deletion Condition, there are indices $1 \leq i < j \leq q+1$ such that

$$s_i ... s_{j+1} = s_{i+1} ... s_j \text{ (2)} \Longleftrightarrow$$
$$s_i ... s_{j+1} s_j ... s_{i+1} = 1 \quad (3)$$

Since we assume (3) is implied by S if it has less than r reflections, take $l(3) < r$. Then the Deletion Condition says that we may omit two factors

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

Since the collection of relations $s_1 \ldots s_r = 1$ in W completely describes W, we will show by induction that
$(S \implies s_{i_1} \ldots s_{i_k} = 1, k < r) \implies (S \implies s_1 \ldots s_r = 1$ (1) for every relation of r reflections in W)
Note: $r = 2q$ for $q \in Z$
Note: The base case of q=1 holds since $s_1 s_2 = 1$ implies that $s_1 = s_2^{-1}$ so $s_1 = s_2$, and realize that $(s_i s_i)^1 = 1$
Now rewrite (1) as $s_1 \ldots s_{q+1} = s_r \ldots s_{q+2}$; since l(right side)=q-1, this means l(left side)=q+1
By the Deletion Condition, there are indices $1 \leq i < j \leq q+1$ such that

$$s_i \ldots s_{j+1} = s_{i+1} \ldots s_j \text{ (2)} \iff$$
$$s_i \ldots s_{j+1} s_j \ldots s_{i+1} = 1 \quad (3)$$

Since we assume (3) is implied by S if it has less than r reflections, take $l(3) < r$. Then the Deletion Condition says that we may omit two factors

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

Since the collection of relations $s_1 ... s_r = 1$ in W completely describes W, we will show by induction that

$(S \implies s_{i_1} ... s_{i_k} = 1, k < r) \implies (S \implies s_1 ... s_r = 1$ (1) for every relation of r reflections in W)

Note: $r = 2q$ for $q \in Z$

Note: The base case of q=1 holds since $s_1 s_2 = 1$ implies that $s_1 = s_2^{-1}$ so $s_1 = s_2$, and realize that $(s_i s_i)^1 = 1$

Now rewrite (1) as $s_1 ... s_{q+1} = s_r ... s_{q+2}$; since l(right side)=q-1, this means l(left side)=q+1

By the Deletion Condition, there are indices $1 \leq i < j \leq q+1$ such that

$$s_i ... s_{j+1} = s_{i+1} ... s_j \text{ (2)} \iff$$
$$s_i ... s_{j+1} s_j ... s_{i+1} = 1 \text{ (3)}$$

Since we assume (3) is implied by S if it has less than r reflections, take $l(3) < r$. Then the Deletion Condition says that we may omit two factors

Proof.

By our induction hypothesis, we can substitute (2) into (1) and we get

$s_1 ... s_i s_{i+1} ... s_{j-1} s_j s_{j+1} ... s_r = s_1 ... s_i (s_i ... s_{j-1}) s_{j+1} ... s_r = s_1 ... s_i ... s_j ... s_r = 1$
which is our desired conclusion

Thus we are done if (3) has less than r reflections

If however (3) has precisely r reflections, say,
$s_1 ... s_q = s_2 ... s_{q+1} \longrightarrow s_1 ... s_q s_{q+1} ... s_2 = 1$ (4), we can rearrange (1) so that

$s_1 ... s_r = 1$ becomes $s_2 ... s_r s_1 = s_1 s_2 ... s_{q+1} ... s_r s_1 = 1$

and then we rearrange this new version of (1) to

$s_2 ... s_{q+2} = s_1 s_r ... s_{q+3}$

We can then repeat the exact argument from before and we will reach a successful conclusion except in case $s_2 ... s_{q+1} = s_3 ... s_{q+2}$ (5) □

Proof.

By our induction hypothesis, we can substitute (2) into (1) and we get

$s_1 ... s_i s_{i+1} ... s_{j-1} s_j s_{j+1} ... s_r = s_1 ... s_i (s_i ... s_{j-1}) s_{j+1} ... s_r = s_1 ... s_i ... s_j ... s_r = 1$
which is our desired conclusion

Thus we are done if (3) has less than r reflections

If however (3) has precisely r reflections, say,
$s_1 ... s_q = s_2 ... s_{q+1} \longrightarrow s_1 ... s_q s_{q+1} ... s_2 = 1$ (4), we can rearrange (1) so that

$s_1 ... s_r = 1$ becomes $s_2 ... s_r s_1 = s_1 s_2 ... s_{q+1} ... s_r s_1 = 1$
and then we rearrange this new version of (1) to
$s_2 ... s_{q+2} = s_1 s_r ... s_{q+3}$
We can then repeat the exact argument from before and we will reach a successful conclusion except in case $s_2 ... s_{q+1} = s_3 ... s_{q+2}$ (5) □

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

By our induction hypothesis, we can substitute (2) into (1) and we get

$s_1 ... s_i s_{i+1} ... s_{j-1} s_j s_{j+1} ... s_r = s_1 ... s_i (s_i ... s_{j-1}) s_{j+1} ... s_r = s_1 ... \hat{s_i} ... \hat{s_j} ... s_r = 1$
which is our desired conclusion
Thus we are done if (3) has less than r reflections
If however (3) has precisely r reflections, say,
$s_1 ... s_q = s_2 ... s_{q+1} \longrightarrow s_1 ... s_q s_{q+1} ... s_2 = 1$ (4), we can rearrange (1) so that
$s_1 ... s_r = 1$ becomes $s_2 ... s_r s_1 = s_1 s_2 ... s_{q+1} ... s_r s_1 = 1$
and then we rearrange this new version of (1) to
$s_2 ... s_{q+2} = s_1 s_r ... s_{q+3}$
We can then repeat the exact argument from before and we will reach a successful conclusion except in case $s_2 ... s_{q+1} = s_3 ... s_{q+2}$ (5) □

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

By our induction hypothesis, we can substitute (2) into (1) and we get

$s_1 ... s_i s_{i+1} ... s_{j-1} s_j s_{j+1} ... s_r = s_1 ... s_i (s_i ... s_{j-1}) s_{j+1} ... s_r = s_1 ... s_i ... s_j ... s_r = 1$
which is our desired conclusion

Thus we are done if (3) has less than r reflections

If however (3) has precisely r reflections, say,

$s_1 ... s_q = s_2 ... s_{q+1} \longrightarrow s_1 ... s_q s_{q+1} ... s_2 = 1$ (4), we can rearrange (1) so that

$s_1 ... s_r = 1$ becomes $s_2 ... s_r s_1 = s_1 s_2 ... s_{q+1} ... s_r s_1 = 1$

and then we rearrange this new version of (1) to

$s_2 ... s_{q+2} = s_1 s_r ... s_{q+3}$

We can then repeat the exact argument from before and we will reach a successful conclusion except in case $s_2 ... s_{q+1} = s_3 ... s_{q+2}$ (5) □

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

By our induction hypothesis, we can substitute (2) into (1) and we get

$s_1 ... s_i s_{i+1} ... s_{j-1} s_j s_{j+1} ... s_r = s_1 ... s_i (s_i ... s_{j-1}) s_{j+1} ... s_r = s_1 ... s_i ... s_j ... s_r = 1$
which is our desired conclusion
Thus we are done if (3) has less than r reflections
If however (3) has precisely r reflections, say,
$s_1 ... s_q = s_2 ... s_{q+1} \longrightarrow s_1 ... s_q s_{q+1} ... s_2 = 1$ (4), we can rearrange (1) so that
$s_1 ... s_r = 1$ becomes $s_2 ... s_r s_1 = s_1 s_2 ... s_{q+1} ... s_r s_1 = 1$
and then we rearrange this new version of (1) to
$s_2 ... s_{q+2} = s_1 s_r ... s_{q+3}$
We can then repeat the exact argument from before and we will reach a successful conclusion except in case $s_2 ... s_{q+1} = s_3 ... s_{q+2}$ (5) □

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

If it happens that $s_2 ... s_{q+1} = s_3 ... s_{q+2}$ (5) and $s_2 ... s_{q+1} = s_1 ... s_q$ (4), then we will try a different strategy

We will rewrite (5) as

$$s_3(s_2 ... s_{q+1}) s_{q+2} ... s_4 = 1$$

and then rearrange it as

$$s_3 s_2 ... s_{q+1} = s_4 ... s_{q+2}$$

We will now repeat the argument from above to show that (5) is a consequence of S by induction, and then substitute (5) into $s_1 ... s_r = 1$ (1) to show that (1) is a consequence of S We find that the only time this fails is when $s_3 s_2 ... s_q = s_2 ... s_{q+1}$

But then $s_1 = s_3$ as before, and we are still at an impasse

Thus we keep on changing indices in (1) like before and continue in the way just described.

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

If it happens that $s_2 ... s_{q+1} = s_3 ... s_{q+2}$ (5) and $s_2 ... s_{q+1} = s_1 ... s_q$ (4), then we will try a different strategy

We will rewrite (5) as

$$s_3(s_2 ... s_{q+1})s_{q+2} ... s_4 = 1$$

and then rearrange it as

$$s_3 s_2 ... s_{q+1} = s_4 ... s_{q+2}$$

We will now repeat the argument from above to show that (5) is a consequence of S by induction, and then substitute (5) into $s_1 ... s_r = 1$ (1) to show that (1) is a consequence of S We find that the only time this fails is when $s_3 s_2 ... s_q = s_2 ... s_{q+1}$

But then $s_1 = s_3$ as before, and we are still at an impasse

Thus we keep on changing indices in (1) like before and continue in the way just described.

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

If it happens that $s_2 ... s_{q+1} = s_3 ... s_{q+2}$ (5) and $s_2 ... s_{q+1} = s_1 ... s_q$ (4), then we will try a different strategy

We will rewrite (5) as

$$s_3 (s_2 ... s_{q+1}) s_{q+2} ... s_4 = 1$$

and then rearrange it as

$$s_3 s_2 ... s_{q+1} = s_4 ... s_{q+2}$$

We will now repeat the argument from above to show that (5) is a consequence of S by induction, and then substitute (5) into $s_1 ... s_r = 1$ (1) to show that (1) is a consequence of S We find that the only time this fails is when $s_3 s_2 ... s_q = s_2 ... s_{q+1}$

But then $s_1 = s_3$ as before, and we are still at an impasse

Thus we keep on changing indices in (1) like before and continue in the way just described.

Proof.

If it happens that $s_2 ... s_{q+1} = s_3 ... s_{q+2}$ (5) and $s_2 ... s_{q+1} = s_1 ... s_q$ (4), then we will try a different strategy
We will rewrite (5) as

$$s_3(s_2 ... s_{q+1}) s_{q+2} ... s_4 = 1$$

and then rearrange it as

$$s_3 s_2 ... s_{q+1} = s_4 ... s_{q+2}$$

We will now repeat the argument from above to show that (5) is a consequence of S by induction, and then substitute (5) into $s_1 ... s_r = 1$ (1) to show that (1) is a consequence of S We find that the only time this fails is when $s_3 s_2 ... s_q = s_2 ... s_{q+1}$
But then $s_1 = s_3$ as before, and we are still at an impasse
Thus we keep on changing indices in (1) like before and continue in the way just described.

□

Proof.

If it happens that $s_2 ... s_{q+1} = s_3 ... s_{q+2}$ (5) and $s_2 ... s_{q+1} = s_1 ... s_q$ (4), then we will try a different strategy

We will rewrite (5) as

$$s_3(s_2 ... s_{q+1})s_{q+2} ... s_4 = 1$$

and then rearrange it as

$$s_3 s_2 ... s_{q+1} = s_4 ... s_{q+2}$$

We will now repeat the argument from above to show that (5) is a consequence of S by induction, and then substitute (5) into $s_1 ... s_r = 1$ (1) to show that (1) is a consequence of S We find that the only time this fails is when $s_3 s_2 ... s_q = s_2 ... s_{q+1}$

But then $s_1 = s_3$ as before, and we are still at an impasse

Thus we keep on changing indices in (1) like before and continue in the way just described.

□

Coxeter Groups $S := \{s_k, k \in \Delta | (s_a s_b)^{m(a,b)} = 1, a, b \in \triangle\}$

Proof.

We find that a successful conclusion is reached except in case

$$s_1 = s_3 = ... = s_{r-1} \text{ and}$$
$$s_2 = s_4 = ... = s_r,$$

But then we may rewrite (1) as

$s_a s_B s_a s_B ... = 1$, which is given by S trivially

□

Parabolic Subgroups

Definition

Fix a simple system Δ in root system R, and let S be the set of simple reflections in W

Take $I \subset S$. Then the reflection group associated with I, W_I, is called a **Parabolic Subgroup** of W and Δ_I is a its associated simple system

Theorem

For a fixed simple system Δ and a corresponding set S of simple reflections. Let $I \subset S$ and define R_I to be the root system corresponding to the reflections of I

Define $W^I := \{w \in W | l(ws) > l(w) \forall s \in I\}$.
Given $w \in W$, there is a unique $u \in W^I$ and a unique $v \in W_I$ such that $w = uv$
Their lengths satisfy $l(w) = l(u) + l(v)$
u is the unique element of smallest length in the coset wW_I

Parabolic Subgroups

Definition

Fix a simple system Δ in root system R, and let S be the set of simple reflections in W
Take $I \subset S$. Then the reflection group associated with I, W_I, is called a **Parabolic Subgroup** of W and Δ_I is a its associated simple system

Theorem

For a fixed simple system Δ and a corresponding set S of simple reflections. Let $I \subset S$ and define R_I to be the root system corresponding to the reflections of I

Define $W^I := \{w \in W | l(ws) > l(w) \forall s \in I\}$.
Given $w \in W$, there is a unique $u \in W^I$ and a unique $v \in W_I$ such that w=uv
Their lengths satisfy l(w)=l(u)+l(v)
u is the unique element of smallest length in the coset wW_I

Parabolic Subgroups

Definition

Fix a simple system Δ in root system R, and let S be the set of simple reflections in W
Take $I \subset S$. Then the reflection group associated with I, W_I, is called a **Parabolic Subgroup** of W and Δ_I is a its associated simple system

Theorem

For a fixed simple system Δ and a corresponding set S of simple reflections. Let $I \subset S$ and define R_I to be the root system corresponding to the reflections of I

Define $W^I := \{w \in W | l(ws) > l(w) \forall s \in I\}$.
Given $w \in W$, there is a unique $u \in W^I$ and a unique $v \in W_I$ such that w=uv
Their lengths satisfy l(w)=l(u)+l(v)
u is the unique element of smallest length in the coset wW_I

Parabolic Subgroups

Definition

Fix a simple system Δ in root system R, and let S be the set of simple reflections in W

Take $I \subset S$. Then the reflection group associated with I, W_I, is called a **Parabolic Subgroup** of W and Δ_I is a its associated simple system

Theorem

For a fixed simple system Δ and a corresponding set S of simple reflections. Let $I \subset S$ and define R_I to be the root system corresponding to the reflections of I

Define $W^I := \{w \in W | l(ws) > l(w) \forall s \in I\}$.

Given $w \in W$, there is a unique $u \in W^I$ and a unique $v \in W_I$ such that w=uv

Their lengths satisfy l(w)=l(u)+l(v)

u is the unique element of smallest length in the coset wW_I

Parabolic Subgroups

Definition

Fix a simple system Δ in root system R, and let S be the set of simple reflections in W
Take $I \subset S$. Then the reflection group associated with I, W_I, is called a **Parabolic Subgroup** of W and Δ_I is a its associated simple system

Theorem

For a fixed simple system Δ and a corresponding set S of simple reflections. Let $I \subset S$ and define R_I to be the root system corresponding to the reflections of I

Define $W^I := \{w \in W | l(ws) > l(w) \forall s \in I\}$.
Given $w \in W$, there is a unique $u \in W^I$ and a unique $v \in W_I$ such that w=uv
Their lengths satisfy l(w)=l(u)+l(v)
u is the unique element of smallest length in the coset wW_I

Parabolic Subgroups

Definition

Fix a simple system Δ in root system R, and let S be the set of simple reflections in W

Take $I \subset S$. Then the reflection group associated with I, W_I, is called a **Parabolic Subgroup** of W and Δ_I is a its associated simple system

Theorem

For a fixed simple system Δ and a corresponding set S of simple reflections. Let $I \subset S$ and define R_I to be the root system corresponding to the reflections of I

Define $W^I := \{w \in W | l(ws) > l(w) \forall s \in I\}$.
Given $w \in W$, there is a unique $u \in W^I$ and a unique $v \in W_I$ such that w=uv
Their lengths satisfy l(w)=l(u)+l(v)
u is the unique element of smallest length in the coset wW_I

Parabolic Subgroups

Definition

Fix a simple system Δ in root system R, and let S be the set of simple reflections in W

Take $I \subset S$. Then the reflection group associated with I, W_I, is called a **Parabolic Subgroup** of W and Δ_I is a its associated simple system

Theorem

For a fixed simple system Δ and a corresponding set S of simple reflections. Let $I \subset S$ and define R_I to be the root system corresponding to the reflections of I

Define $W^I := \{w \in W | l(ws) > l(w) \forall s \in I\}$.

Given $w \in W$, there is a unique $u \in W^I$ and a unique $v \in W_I$ such that $w=uv$

Their lengths satisfy $l(w)=l(u)+l(v)$

u is the unique element of smallest length in the coset wW_I

Parabolic Subgroups

Definition

Fix a simple system Δ in root system R, and let S be the set of simple reflections in W
Take $I \subset S$. Then the reflection group associated with I, W_I, is called a **Parabolic Subgroup** of W and Δ_I is a its associated simple system

Theorem

For a fixed simple system Δ and a corresponding set S of simple reflections. Let $I \subset S$ and define R_I to be the root system corresponding to the reflections of I

Define $W^I := \{w \in W | l(ws) > l(w) \forall s \in I\}$.
Given $w \in W$, there is a unique $u \in W^I$ and a unique $v \in W_I$ such that $w{=}uv$
Their lengths satisfy $l(w){=}l(u){+}l(v)$
u is the unique element of smallest length in the coset wW_I

Parabolic Subgroups

Proof.

Given a reduced $w \in W$, choose a representative of wW_I called u of smallest length and choose $v \in W_I$ such that w=uv and v is reduced.

To build u, we take w and remove every element of W_I that we can from w to create a reduced expression , which also implies that $l(us) > l(s) \forall s \in W_I$.Thus $u \in W^I$

We know $l(w) \leq l(u) + l(v)$, but u,v are both reduced:
Removing a factor from u yields an element smaller than u,and v is reduced by assumption.
Also, u and v come from disjoint subsets of W
Therefore, $l(w) = l(uv) = l(u) + l(v)$

If u is not unique, ∃u' such that $u' > u$
But $u' > u \implies \exists s_i \in wW_I$ such that $l(u's) < l(u')$ contradicting $u' \in W^I$, so u' cannot exist

Parabolic Subgroups

Proof.

Given a reduced $w \in W$, choose a representative of wW_I called u of smallest length and choose $v \in W_I$ such that w=uv and v is reduced.
To build u, we take w and remove every element of W_I that we can from w to create a reduced expression , which also implies that $l(us) > l(s) \forall s \in W_I$.Thus $u \in W^I$

We know $l(w) \leq l(u) + l(v)$, but u,v are both reduced:
Removing a factor from u yields an element smaller than u,and v is reduced by assumption.
Also, u and v come from disjoint subsets of W
Therefore, $l(w) = l(uv) = l(u) + l(v)$

If u is not unique, ∃u' such that $u' > u$
But $u' > u \implies \exists s_i \in wW_I$ such that $l(u's) < l(u')$ contradicting $u' \in W^I$, so u' cannot exist

Parabolic Subgroups

Proof.

Given a reduced $w \in W$, choose a representative of wW_I called u of smallest length and choose $v \in W_I$ such that w=uv and v is reduced. To build u, we take w and remove every element of W_I that we can from w to create a reduced expression , which also implies that $l(us) > l(s) \forall s \in W_I$. Thus $u \in W^I$

We know $l(w) \leq l(u) + l(v)$, but u,v are both reduced:
Removing a factor from u yields an element smaller than u,and v is reduced by assumption.
Also, u and v come from disjoint subsets of W
Therefore, $l(w) = l(uv) = l(u) + l(v)$

If u is not unique, ∃u' such that $u' > u$
But $u' > u \implies \exists s_i \in wW_I$ such that $l(u's) < l(u')$ contradicting $u' \in W^I$, so u' cannot exist

Parabolic Subgroups

Proof.

Given a reduced $w \in W$, choose a representative of wW_I called u of smallest length and choose $v \in W_I$ such that w=uv and v is reduced.
To build u, we take w and remove every element of W_I that we can from w to create a reduced expression , which also implies that $l(us) > l(s) \forall s \in W_I$.Thus $u \in W^I$

We know $l(w) \leq l(u) + l(v)$, but u,v are both reduced:
Removing a factor from u yields an element smaller than u,and v is reduced by assumption.
Also, u and v come from disjoint subsets of W
Therefore, $l(w) = l(uv) = l(u) + l(v)$

If u is not unique, ∃u' such that $u' > u$
But $u' > u \implies \exists s_i \in wW_I$ such that $l(u's) < l(u')$ contradicting $u' \in W^I$, so u' cannot exist

Parabolic Subgroups

Proof.

Given a reduced $w \in W$, choose a representative of wW_I called u of smallest length and choose $v \in W_I$ such that w=uv and v is reduced. To build u, we take w and remove every element of W_I that we can from w to create a reduced expression , which also implies that $l(us) > l(s) \forall s \in W_I$.Thus $u \in W^I$

We know $l(w) \leq l(u) + l(v)$, but u,v are both reduced:
Removing a factor from u yields an element smaller than u,and v is reduced by assumption.
Also, u and v come from disjoint subsets of W
Therefore, $l(w) = l(uv) = l(u) + l(v)$

If u is not unique, ∃u' such that $u' > u$
But $u' > u \implies \exists s_i \in wW_I$ such that $l(u's) < l(u')$ contradicting $u' \in W^I$, so u' cannot exist

Parabolic Subgroups

Proof.

Given a reduced $w \in W$, choose a representative of wW_I called u of smallest length and choose $v \in W_I$ such that w=uv and v is reduced. To build u, we take w and remove every element of W_I that we can from w to create a reduced expression , which also implies that $l(us) > l(s) \forall s \in W_I$.Thus $u \in W^I$

We know $l(w) \leq l(u) + l(v)$, but u,v are both reduced:
Removing a factor from u yields an element smaller than u,and v is reduced by assumption.
Also, u and v come from disjoint subsets of W
Therefore, $l(w) = l(uv) = l(u) + l(v)$

If u is not unique, ∃u' such that $u' > u$
But $u' > u \implies \exists s_i \in wW_I$ such that $l(u's) < l(u')$ contradicting $u' \in W^I$, so u' cannot exist

Parabolic Subgroups

Proof.

Given a reduced $w \in W$, choose a representative of wW_I called u of smallest length and choose $v \in W_I$ such that w=uv and v is reduced. To build u, we take w and remove every element of W_I that we can from w to create a reduced expression , which also implies that $l(us) > l(s) \forall s \in W_I$.Thus $u \in W^I$

We know $l(w) \leq l(u) + l(v)$, but u,v are both reduced:
Removing a factor from u yields an element smaller than u,and v is reduced by assumption.
Also, u and v come from disjoint subsets of W
Therefore, $l(w) = l(uv) = l(u) + l(v)$

If u is not unique, ∃u' such that $u' > u$
But $u' > u \implies \exists s_i \in wW_I$ such that $l(u's) < l(u')$ contradicting $u' \in W^I$, so u' cannot exist

Parabolic Subgroups

Proof.

Given a reduced $w \in W$, choose a representative of wW_I called u of smallest length and choose $v \in W_I$ such that w=uv and v is reduced. To build u, we take w and remove every element of W_I that we can from w to create a reduced expression , which also implies that $l(us) > l(s) \forall s \in W_I$.Thus $u \in W^I$

We know $l(w) \leq l(u) + l(v)$, but u,v are both reduced:
Removing a factor from u yields an element smaller than u,and v is reduced by assumption.
Also, u and v come from disjoint subsets of W
Therefore, $l(w) = l(uv) = l(u) + l(v)$

If u is not unique, ∃u' such that $u' > u$
But $u' > u \implies \exists s_i \in wW_I$ such that $l(u's) < l(u')$ contradicting $u' \in W^I$, so u' cannot exist

Parabolic Subgroups

Proof.

Given a reduced $w \in W$, choose a representative of wW_I called u of smallest length and choose $v \in W_I$ such that w=uv and v is reduced.
To build u, we take w and remove every element of W_I that we can from w to create a reduced expression , which also implies that $l(us) > l(s) \forall s \in W_I$.Thus $u \in W^I$

We know $l(w) \leq l(u) + l(v)$, but u,v are both reduced:
Removing a factor from u yields an element smaller than u,and v is reduced by assumption.
Also, u and v come from disjoint subsets of W
Therefore, $l(w) = l(uv) = l(u) + l(v)$

If u is not unique, ∃u' such that $u' > u$
But $u' > u \implies \exists s_i \in wW_I$ such that $l(u's) < l(u')$ contradicting $u' \in W^I$, so u' cannot exist

Parabolic Subgroups

Proof.

Given a reduced $w \in W$, choose a representative of wW_I called u of smallest length and choose $v \in W_I$ such that w=uv and v is reduced. To build u, we take w and remove every element of W_I that we can from w to create a reduced expression , which also implies that $l(us) > l(s) \forall s \in W_I$.Thus $u \in W^I$

We know $l(w) \leq l(u) + l(v)$, but u,v are both reduced:
Removing a factor from u yields an element smaller than u,and v is reduced by assumption.
Also, u and v come from disjoint subsets of W
Therefore, $l(w) = l(uv) = l(u) + l(v)$

If u is not unique, ∃u' such that $u' > u$
But $u' > u \implies \exists s_i \in wW_I$ such that $l(u's) < l(u')$ contradicting $u' \in W^I$, so u' cannot exist

Poincare Polynomials

Definition

A **Poincare Polynomial** is a polynomial of indeterminate t that is a bookkeeper for the elements of a reflection group W

Define a sequence

$$a_n := Card\{w \in W \; | l(w) = n\}$$

Then the **Poincare Polynomial** for W is

$$\mathrm{W(t)} := \sum_{n \geq 0} a_n t^n = \sum_{w \in W} t^{l(w)}$$

Example

Take the reflection group $W = S_3$
We see that $W(t) = 1 + 2t + 2t^2 + t^3$ since
$W = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

Poincare Polynomials

Definition

A **Poincare Polynomial** is a polynomial of indeterminate t that is a bookkeeper for the elements of a reflection group W

Define a sequence

$$a_n := Card\{w \in W \mid l(w) = n\}$$

Then the **Poincare Polynomial** for W is

$$W(t) := \sum_{n \geq 0} a_n t^n = \sum_{w \in W} t^{l(w)}$$

Example

Take the reflection group $W = S_3$

We see that $W(t) = 1 + 2t + 2t^2 + t^3$ since $W = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

Poincare Polynomials

Definition

A **Poincare Polynomial** is a polynomial of indeterminate t that is a bookkeeper for the elements of a reflection group W

Define a sequence

$$a_n := Card\{w \in W \ |l(w) = n\}$$

Then the **Poincare Polynomial** for W is

$$W(t) := \sum_{n \geq 0} a_n t^n = \sum_{w \in W} t^{l(w)}$$

Example

Take the reflection group $W = S_3$

We see that $W(t) = 1 + 2t + 2t^2 + t^3$ since

$W = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

Poincare Polynomials

Definition

A **Poincare Polynomial** is a polynomial of indeterminate t that is a bookkeeper for the elements of a reflection group W

Define a sequence

$$a_n := Card\{w \in W \,| l(w) = n\}$$

Then the **Poincare Polynomial** for W is

$$\mathrm{W(t)} := \sum_{n \geq 0} a_n t^n = \sum_{w \in W} t^{l(w)}$$

Example

Take the reflection group $W = S_3$

We see that $W(t) = 1 + 2t + 2t^2 + t^3$ since

$W = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

Poincare Polynomials

Definition

A **Poincare Polynomial** is a polynomial of indeterminate t that is a bookkeeper for the elements of a reflection group W

Define a sequence

$$a_n := Card\{w \in W \mid l(w) = n\}$$

Then the **Poincare Polynomial** for W is

$$\mathrm{W(t)} := \sum_{n\geq 0} a_n t^n = \sum_{w \in W} t^{l(w)}$$

Example

Take the reflection group $W = S_3$
We see that $W(t) = 1 + 2t + 2t^2 + t^3$ since
$W = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

Poincare Polynomials

Theorem

Since $W(t) = W^I(t)W_I(t)$*, we can show that*

$$\sum_{I\subset S}(-1)^I \frac{W(t)}{W_I(t)} = \sum_{I\subset S}(-1)^I W^I(t) = t^N,$$

where N=l(w_0*), the longest element of W* $(ie : l(w_0 s) \leq l(w_0) \forall s \in W)$

Example

The theorem can be proven combinatorically,but let us see how it works for S_3,

$W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$

$I = \{s_1\} \implies term1 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_2\} \implies term2 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_1, s_2\} \implies term3 : (-1)^2(t^0)$

$I = \phi \implies term4 : (-1)^0(t^0 + 2t^1 + 2t^2 + t^3)$

Adding these 4 terms together equals t^3 as desired

Poincare Polynomials

Theorem

Since $W(t) = W^I(t)W_I(t)$, *we can show that*

$$\sum_{I\subset S}(-1)^I \frac{W(t)}{W_I(t)} = \sum_{I\subset S}(-1)^I W^I(t) = t^N,$$

where N=l(w_0*), the longest element of W (ie :* $l(w_0 s) \leq l(w_0)\forall s \in W$*)*

Example

The theorem can be proven combinatorically,but let us see how it works for S_3,

$W = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

$I = \{s_1\} \implies term1 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_2\} \implies term2 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_1, s_2\} \implies term3 : (-1)^2(t^0)$

$I = \phi \implies term4 : (-1)^0(t^0 + 2t^1 + 2t^2 + t^3)$

Adding these 4 terms together equals t^3 as desired

Poincare Polynomials

Theorem

Since $W(t) = W^I(t)W_I(t)$*, we can show that*

$$\sum_{I \subset S}(-1)^I \frac{W(t)}{W_I(t)} = \sum_{I \subset S}(-1)^I W^I(t) = t^N,$$

where $N = l(w_0)$*, the longest element of* W *(ie :* $l(w_0 s) \leq l(w_0) \forall s \in W$*)*

Example

The theorem can be proven combinatorically, but let us see how it works for S_3,

$W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$

$I = \{s_1\} \implies term1 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_2\} \implies term2 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_1, s_2\} \implies term3 : (-1)^2(t^0)$

$I = \phi \implies term4 : (-1)^0(t^0 + 2t^1 + 2t^2 + t^3)$

Adding these 4 terms together equals t^3 as desired

Poincare Polynomials

Theorem

Since $W(t) = W^I(t)W_I(t)$, *we can show that*

$$\sum_{I \subset S}(-1)^I \frac{W(t)}{W_I(t)} = \sum_{I \subset S}(-1)^I W^I(t) = t^N,$$

where $N = l(w_0)$, *the longest element of* W (*ie* : $l(w_0 s) \leq l(w_0) \forall s \in W$)

Example

The theorem can be proven combinatorically,but let us see how it works for S_3,

$W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$

$I = \{s_1\} \implies term1 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_2\} \implies term2 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_1, s_2\} \implies term3 : (-1)^2(t^0)$

$I = \phi \implies term4 : (-1)^0(t^0 + 2t^1 + 2t^2 + t^3)$

Adding these 4 terms together equals t^3 as desired

Poincare Polynomials

Theorem

Since $W(t) = W^I(t)W_I(t)$, *we can show that*

$$\sum_{I\subset S}(-1)^I \frac{W(t)}{W_I(t)} = \sum_{I\subset S}(-1)^I W^I(t) = t^N,$$

where N=l(w_0*), the longest element of W (ie :* $l(w_0 s) \leq l(w_0) \forall s \in W$*)*

Example

The theorem can be proven combinatorically,but let us see how it works for S_3,

$W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$

$I = \{s_1\} \Longrightarrow term1 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_2\} \Longrightarrow term2 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_1, s_2\} \Longrightarrow term3 : (-1)^2(t^0)$

$I = \phi \Longrightarrow term4 : (-1)^0(t^0 + 2t^1 + 2t^2 + t^3)$

Adding these 4 terms together equals t^3 as desired

Poincare Polynomials

Theorem

Since $W(t) = W^I(t)W_I(t)$, *we can show that*

$$\sum_{I \subset S}(-1)^I \frac{W(t)}{W_I(t)} = \sum_{I \subset S}(-1)^I W^I(t) = t^N,$$

where N=l(w_0*), the longest element of W (ie :* $l(w_0 s) \leq l(w_0) \forall s \in W$*)*

Example

The theorem can be proven combinatorically,but let us see how it works for S_3,

$W = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

$I = \{s_1\} \implies term1 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_2\} \implies term2 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_1, s_2\} \implies term3 : (-1)^2(t^0)$

$I = \phi \implies term4 : (-1)^0(t^0 + 2t^1 + 2t^2 + t^3)$

Adding these 4 terms together equals t^3 as desired

Poincare Polynomials

Theorem

Since $W(t) = W^I(t)W_I(t)$, *we can show that*

$$\sum_{I \subset S} (-1)^I \frac{W(t)}{W_I(t)} = \sum_{I \subset S} (-1)^I W^I(t) = t^N,$$

where $N = l(w_0)$, *the longest element of* W (*ie* : $l(w_0 s) \leq l(w_0) \forall s \in W$)

Example

The theorem can be proven combinatorically,but let us see how it works for S_3,

$W = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

$I = \{s_1\} \implies term1 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_2\} \implies term2 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_1, s_2\} \implies term3 : (-1)^2(t^0)$

$I = \phi \implies term4 : (-1)^0(t^0 + 2t^1 + 2t^2 + t^3)$

Adding these 4 terms together equals t^3 as desired

Theorem

Since $W(t) = W^I(t)W_I(t)$, *we can show that*

$$\sum_{I\subset S}(-1)^I \frac{W(t)}{W_I(t)} = \sum_{I\subset S}(-1)^I W^I(t) = t^N,$$

where $N = l(w_0)$, *the longest element of* W (*ie* : $l(w_0 s) \leq l(w_0) \forall s \in W$)

Example

The theorem can be proven combinatorically,but let us see how it works for S_3,

$W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$

$I = \{s_1\} \implies term1 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_2\} \implies term2 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_1, s_2\} \implies term3 : (-1)^2(t^0)$

$I = \phi \implies term4 : (-1)^0(t^0 + 2t^1 + 2t^2 + t^3)$

Adding these 4 terms together equals t^3 as desired

Poincare Polynomials

Theorem

Since $W(t) = W^I(t)W_I(t)$, *we can show that*

$$\sum_{I\subset S}(-1)^I \frac{W(t)}{W_I(t)} = \sum_{I\subset S}(-1)^I W^I(t) = t^N,$$

where $N=l(w_0)$, *the longest element of* W *(ie :* $l(w_0 s) \leq l(w_0) \forall s \in W$*)*

Example

The theorem can be proven combinatorically,but let us see how it works for S_3,

$W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$

$I = \{s_1\} \Longrightarrow term1 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_2\} \Longrightarrow term2 : (-1)^1(t^0 + t^1 + t^2)$

$I = \{s_1, s_2\} \Longrightarrow term3 : (-1)^2(t^0)$

$I = \phi \Longrightarrow term4 : (-1)^0(t^0 + 2t^1 + 2t^2 + t^3)$

Adding these 4 terms together equals t^3 as desired