The Heisenberg Category

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Category

Definition

A *category*, $\mathcal{C}$, consists of

- A class of *objects*, denoted $Ob(\mathcal{C})$
- A class of *morphisms* between the objects of $\mathcal{C}$. If $X, Y \in Ob(\mathcal{C})$, then we denote the class of morphisms from X to Y by $Hom_{\mathcal{C}}(X, Y)$
- We can compose morphisms whenever it makes sense. Composition of f and g is denoted $f \circ g$ (if this makes sense).

Additionally, we impose the following relations:

- (*associativity*): whenever compositions of morphisms make sense, we have $f \circ (g \circ h) = (f \circ g) \circ h$
- (*identity*) for any object $x \in \mathcal{C}$, there is a morphism $1_x : x \to x$ such that for any morphism $f : x \to y$ and $g : a \to x$ in $\mathcal{C}$, we have $1_x \circ g = g$ and $f \circ 1_x = f$.

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Example

The category **Set** is the category whose objects are sets and whose morphisms are functions between sets. Composition of morphisms is defined as the composition of functions.

Example

The category $\Bbbk$**-Vect** is the category whose objects are vector spaces over a fixed field $\Bbbk$ and whose morphisms are linear transformations between these vector spaces. Composition of morphisms is defined as composition of linear transformations.

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Strict $\mathbb{k}$-Linear Monoidal Category

Definition

A *strict monoidal category* is a category $\mathcal{C}$ together with

- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbf{1}$,

such that for all $A, B, C \in Ob(\mathcal{C})$,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, and
- $\mathbf{1} \otimes A = A = A \otimes \mathbf{1}$

Definition

Let $\mathbb{k}$ be a commutative ring. A strict $\mathbb{k}$-*linear* monoidal category is a strict monoidal category such that

- each morphism space is a $\mathbb{k}$-module, and
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Strict Monoidal Categories

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The category of sets, **Set**, is a monoidal category with the Cartesian product as the bifunctor and any one-element set as the unit object.

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The category of vector spaces over a fixed field $\mathbb{k}$, $\mathbb{k}$-**Vect**, is a monoidal category with the usual tensor product, and the field $\mathbb{k}$ serving as the unit object.

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The category of endofunctors from a category $\mathcal{C}$ to itself is a *strict* monoidal category with composition of functors as the tensor product, and the identity functor as the unit object.

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String Diagrams 1

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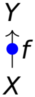

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The identity on X is given by a morphism

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Additionally, we may write morphisms $f : \mathbf{1} \to X \otimes Y$ and $g : X \otimes Y \to \mathbf{1}$ as

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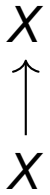

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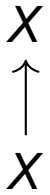

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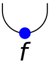

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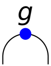

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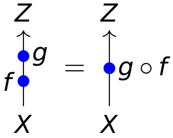
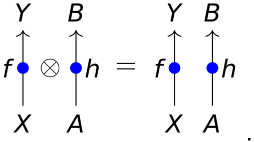
We compose morphisms vertically, and we tensor morphisms horizontally. So, if $f : X \to Y$, $g : Y \to Z$, and $h : A \to B$, we write





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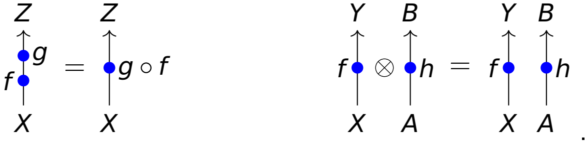

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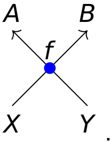

Example

Consider a strict $\Bbbk$-linear monoidal category, $\mathcal{S}$, defined as follows. The objects of $\mathcal{S}$ are generated by a single object, Q_+. That is, the objects are $\mathbf{1}, Q_+, Q_+Q_+, \dots,$ where juxtaposition denotes tensor product. The morphisms are generated by

 $: Q_+Q_+ \to Q_+Q_+$

and are subject to the relations

 (1) (2)

Then

$$\mathrm{End}_{\mathcal{S}}(Q_+^{\otimes n}) = \Bbbk S_n,$$

where S_n is the symmetric group on n letters.

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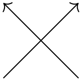 $: Q_+Q_+ \to Q_+Q_+$

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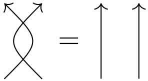
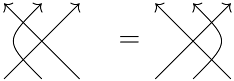
 (1)  (2)

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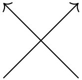

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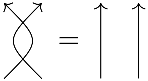

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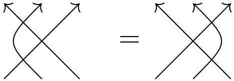

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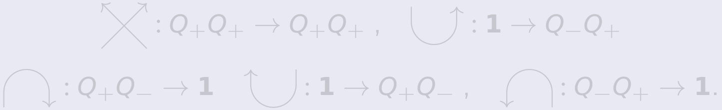
The category $\mathcal{H}'$ is the strict $\Bbbk$-linear monoidal category defined as follows. The objects are generated by objects Q_+ and Q_-, where we use juxtaposition to denote tensor product. For example, Q_+Q_- means $Q_+ \otimes Q_-$. The morphisms are generated by



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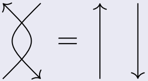
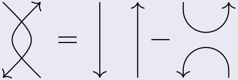
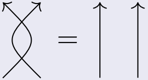
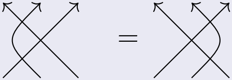
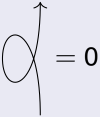
 (3)

 (4)

 (5)

 (6)



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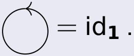

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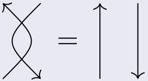
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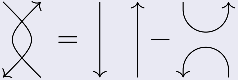
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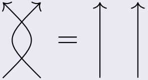
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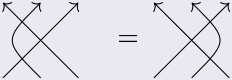
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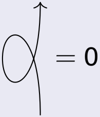
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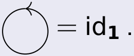
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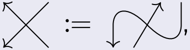
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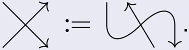
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Additive Envelope

Let $\mathcal{C}$ be a $\Bbbk$-linear monoidal category.

The *additive envelope* of $\mathcal{C}$ is a category whose

- objects are formal finite direct sums $\bigoplus_{i=1}^{n} X_i$ of objects $X_i \in \mathcal{C}$,
- morphisms

$$f : \bigoplus_{i=1}^{n} X_i \to \bigoplus_{j=1}^{m} Y_j$$

are $m \times n$ matrices whose (j, i)—entry is a morphism

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The *additive envelope* of $\mathcal{C}$ is a category whose

- objects are formal finite direct sums $\bigoplus_{i=1}^{n} X_i$ of objects $X_i \in \mathcal{C}$,
- morphisms

$$f : \bigoplus_{i=1}^{n} X_i \to \bigoplus_{j=1}^{m} Y_j$$

are $m \times n$ matrices whose (j, i)—entry is a morphism

$$f_{i,j} : X_i \to Y_j.$$

Composition of morphisms is given by matrix multiplication.

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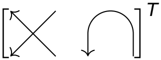
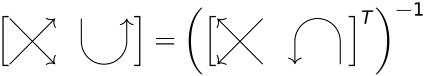
Isomorphism

Consider the morphism



$$: Q_- Q_+ \to Q_+ Q_- \oplus \mathbf{1}. \qquad (11)$$

We claim that



$$: Q_+ Q_- \oplus \mathbf{1} \to Q_- Q_+ \qquad (12)$$

Composing matrices in one direction gives the following relation, which must hold in $\mathcal{H}'$:

(13)

The relation (13) follows from the definition of $\mathcal{H}'$.

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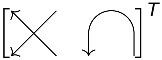

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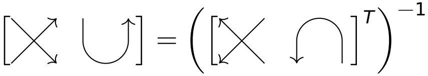

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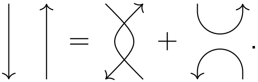

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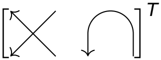

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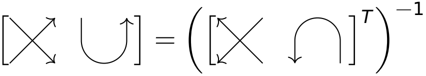

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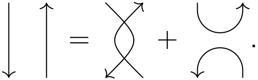

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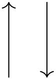
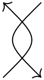
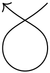
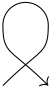
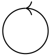
 $=$  (14)

 $= 0,$ (15)

 $= 0,$ (16)

 $= \mathrm{id}_{\mathbf{1}},$ (17)

The relations (14) and (17) follow from the definition of $\mathcal{H}'$. The relation (15) follows from the calculation

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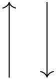
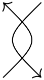
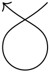
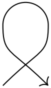
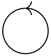
 $=$  (14)

 $= 0,$ (15)

 $= 0,$ (16)

 $= \mathrm{id}_{\mathbf{1}},$ (17)

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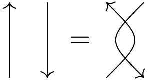

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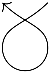

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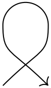

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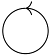

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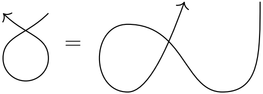

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The Heisenberg Algebra

Definition

The one-variable Heisenberg algebra is the associative unital $\mathbb{C}$-algebra with generators p and q subject to the *canonical commutation relation*:

$$\mathbf{pq} = \mathbf{qp} + \mathbf{1}.$$

Recall the isomorphism in the additive envelope of $\mathcal{H}'$:

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