Twisted Formal Group Algebras

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August 16, 2017

Reflection Group - Definition and Example

Definition

Let V be a vector space. Let t, α be vectors in V. We call s_α a reflection if

$$s_\alpha(t) = t - 2\frac{(t,\alpha)}{(\alpha,\alpha)}\alpha$$

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A group that is generated by reflections is called a **Reflection Group**.

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Let V be a vector space. A **Root System**, Σ, of V is a set of vectors of V such that for all $a \in \Sigma$:

1. $\Sigma \cap \mathbf{c}a = \{a, -a\}, \mathbf{c} \in \mathbf{R}$
2. $s_a\Sigma = \Sigma$

Every root system Σ is associated with some reflection group W.

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Positive, Negative, and Simple Systems

Definition

Let V be an vector space. Let $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$ be a Root System of V. Let $\Delta = \{\beta_1, \ldots, \beta_k\}$, $k \leq n$. Δ is a **Simple System** of Φ if it is a vector space basis of V and for all $\alpha_j \in \Sigma$

$$\alpha_j = \sum_{i=1}^{k} a_i \beta_i,$$

where, for each j, $a_i > 0$ or $a_i < 0$ for all i.

Those α_j for which all $a_i > 0$ are called **Positive Roots**. The set of these roots is called a **Positive System**. Those α_j for which all $a_i < 0$ are called **Negative Roots**. The set of these roots is called a **Negative System**.

The reflections associated with simple roots are called **Simple Reflections**.

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Coxeter Group

Theorem

Let Σ be a root system in V, and let Δ be a simple system of Σ. Let W be the reflection group associated to Σ. Then W is generated by simple reflections in the following way:

$$W = \langle s_\alpha, s_\beta, \alpha, \beta \in \Delta | (s_\alpha s_\beta)^{m_{\alpha,\beta}} = 1 \rangle$$

where $m_{\alpha,\beta}$ is the order of $s_\alpha s_\beta$ in W.

Example

Suppose $\Sigma = \{\pm(1,0), \pm(1,1), \pm(0,1), \pm(-1,1)\}$. Then a simple system of Σ is $\Delta = \{s_{(1,0)}, s_{(-1,1)}\}$. Let $1 = (1,0)$, $2 = (-1,1)$. Then from our Coxeter Relations: $W = \{1, s_1, s_2, s_{12}, s_{21}, s_{121}, s_{212}, s_{1212}\}$

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Formal Group Law

Definition

Let R be a commutative ring.
Let $F(x,y) = x + y + \sum_{i,j} a_{ij} x^i y^j$, $a_{ij} \in R$, be a formal power series in variables x and y.
(F, R) is a **Formal Group Law** (FGL) if

1. $F(x,0) = F(0,x)$
2. $F(x,y) = F(y,x)$
3. $F(F(x,y),z) = F(x,F(y,z))$

Notation: $F(x,y) = x +_F y$

Every FGL has a "formal inverse" denoted $-_F x$ with the property

$$x +_F (-_F x) = 0.$$

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Examples

Example

Let F be the additive FGL. Then

$$F(x, y) = x + y$$

Here, $-_F x = -x$

Example

Let (F, R) be the multiplicative FGL, and let $\beta \in R$, $\beta \neq 0$. Then

$$F(x, y) = x + y - \beta xy$$

Here, $-_F x = \frac{x}{\beta x - 1} = -x(1 + \beta x + (\beta x)^2 + \ldots) = -x - (\beta x)^2 - (\beta x)^3 - \ldots$

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Let (F, R) be the Lorentz FGL, and let $\beta \in R$, $\beta \neq 0$. Then

$$F(x,y) = \frac{x+y}{1+\beta xy} = (x+y)(1+(-\beta xy)+(-\beta xy)^2+...)$$

Here, $-_F x = -x$

Example

Let $(F, \mathbb{L})$ be the universal FGL. Then

$$F(x,y) = x + y + \sum_{i,j} a_{ij} x^i y^j$$

where the variables a_{ij} are resticted by the associativity and commutativity of the FGL. They lie in the Lazard ring $\mathbb{L}$.

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Formal Group Algebra

Let (F, R) be an FGL, and suppose Λ is a finite abelian group. Let $R[[x_\lambda]] = R[[\{x_\lambda | \lambda \in \Lambda\}]]$ be a formal power series over R in variables x_λ indexed by $\lambda \in \Lambda$.

Let J_F be the smallest ideal generated by the elements

$$x_0 \text{ and } x_{\lambda_1} +_F x_{\lambda_2} - x_{\lambda_1+\lambda_2}$$

Definition

The quotient is called the **Formal Group Ring/Algebra** of Λ with respect to (F, R):

$$R[[\Lambda]]_F = R[[x_\lambda]]/J_F$$

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Twisted Formal Group Algebra

Let Σ be a finite root system associated with reflection group W. Let Λ be a certain abelian group such that $\Sigma \subset \Lambda$.

Denote by $Q_F = Q^{(R,F)}$ the field of fractions generated by $R[[\Lambda]]_F$ and $\{x_\lambda^{-1} | \lambda \in \Lambda \backslash \{0\}\}$. W acts on a polynomial $f(x_{\alpha_1}, ..., x_{\alpha_n}) \in Q_F$ in the following way:

$$w(f(x_{\alpha_1}, ..., x_{\alpha_n})) = f(x_{w(\alpha_1)}, ..., x_{w(\alpha_n)})$$

Definition

The left R-module $Q_W := Q_F \otimes_R R[W]$ is the **Twisted Formal Group Algebra**, with multiplication given by

$$(q_1\delta_{w_1})(q_2\delta_{w_2}) = (q_1 w_1(q_2))(\delta_{w_1}\delta_{w_2})$$

Here, $q_i \in Q_F$ and δ_{w_i} denotes an element of Q_W: it is not the same as the reflection w_i.

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Denote by $Q_F = Q^{(R,F)}$ the field of fractions generated by $R[[\Lambda]]_F$ and $\{x_\lambda^{-1} | \lambda \in \Lambda \backslash \{0\}\}$. W acts on a polynomial $f(x_{\alpha_1}, ..., x_{\alpha_n}) \in Q_F$ in the following way:

$$w(f(x_{\alpha_1}, ..., x_{\alpha_n})) = f(x_{w(\alpha_1)}, ..., x_{w(\alpha_n)})$$

Definition

The left R-module $Q_W := Q_F \otimes_R R[W]$ is the **Twisted Formal Group Algebra**, with multiplication given by

$$(q_1\delta_{w_1})(q_2\delta_{w_2}) = (q_1 w_1(q_2))(\delta_{w_1}\delta_{w_2})$$

Here, $q_i \in Q_F$ and δ_{w_i} denotes an element of Q_W: it is not the same as the reflection w_i.

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Formal Demazure Algebra

Definition

Let $\alpha \in \Sigma$. The **Formal Demazure Element** of α with respect to the FGL, (F, R), is defined as

$$X_\alpha^F = \frac{1}{x_\alpha}(1 - \delta_{s_\alpha})$$

Definition

The **Formal Demazure Algebra,** D_W^F, is the R-subalgebra of Q_W generated by the elements X_α.

The **Formal Affine Demazure Algebra**, $\mathbf{D}_\mathbf{W}^\mathbf{F}$, is the R-subalgebra of Q_W generated by $R[[\Lambda]]$ and the elements X_α

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Question

Question: How can we describe D_W^F when $W = I_2(5)$ is the dihedral group of order 5, and F is an arbitrary FGL?

Let Σ be the root system corresponding to $I_2(5)$. Fix FGL (F, R).

Notations/Assumptions:

- For $\lambda \in \Sigma$, $\lambda = \pm\alpha_i$ for some $i = 1, ..., 5$
- Denote by $x_{-\alpha_i}$, $i = 1, ..., 5$, the formal inverse of x_{α_i} under (F, R)
- Denote by α_1 and α_2 our simple roots of Σ
- Set $X_{\alpha_i} = X_i$ for $i = 1, ..., 5$

The following slide shows the positions and labelling of the roots that will be used from now on.

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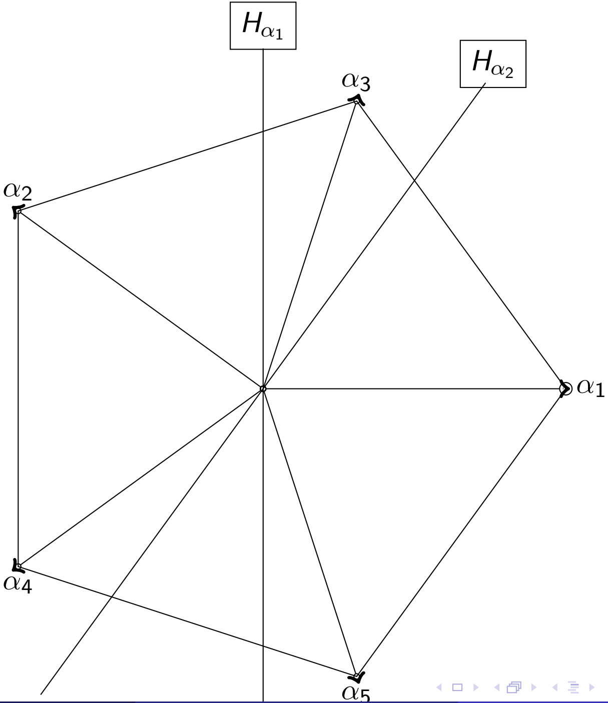
H_{α_1}
H_{α_2}
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α_2
α_1
α_4
α_5

Notation: Let $s_{\alpha_i} = s_i$ and $\alpha_i = i$. The following table represents the permuatation of roots obtained after applying a reflection of $I_2(5)$ to the roots of Σ

1	1	−1	2	−2	3	−3	4	−4	5	−5
s_1	−1	1	-4	4	−5	5	−2	2	−3	3
s_2	−5	5	−2	2	-4	4	−3	3	−1	1
s_{12}	3	−3	4	−4	2	−2	5	−5	1	−1
s_{21}	5	−5	3	−3	1	−1	2	−2	4	−4
s_{121}	−3	3	−5	5	−1	1	−4	4	−2	2
s_{212}	−4	4	−3	3	−2	2	−1	1	−5	5
s_{1212}	2	−2	5	−5	4	−4	1	−1	3	−3
s_{2121}	4	−4	1	−1	5	−5	3	−3	2	−2
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s_{12}	3	−3	4	−4	2	−2	5	−5	1	−1
s_{21}	5	−5	3	−3	1	−1	2	−2	4	−4
s_{121}	−3	3	−5	5	−1	1	−4	4	−2	2
s_{212}	−4	4	−3	3	−2	2	−1	1	−5	5
s_{1212}	2	−2	5	−5	4	−4	1	−1	3	−3
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s_{12121}	−2	2	−1	1	−3	3	−5	5	−4	4

Generating $D^F_{I_2(5)}$

Theorem

Set $s_j s_k = s_{jk}$. Let $I_2(m) = I_2(5)$. Then

$$X_{21212} - X_{12121} = (k^3_{\alpha_2} X_{212} + s_{12121}(k^3_{-\alpha_2})X_{121}) + (k^2_{\alpha_2} X_{12} + s_{12121}(k^2_{-\alpha_2})X_{21}) + (k^1_{\alpha_2})X_2 + s_{12121}(k^1_{-\alpha_2})X_1)$$

where $k^i_{-\alpha_2}$ is obtained from $k^i_{\alpha_2}$ by applying the map $x_{\pm i} \mapsto x_{\mp i}$, and

$$k^3_{\alpha_2} = \frac{1}{x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} x_{(-\alpha_5)}} - \frac{1}{x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} x_{\alpha_4}} - \frac{1}{x_{(-\alpha_2)} x_{(-\alpha_3)} x_{(-\alpha_4)} x_{(-\alpha_5)}} + \frac{1}{x_{\alpha_1} x_{(-\alpha_2)} x_{(-\alpha_4)} x_{(-\alpha_5)}} - \frac{1}{x_{\alpha_1} x_{\alpha_3} x_{(-\alpha_4)} x_{(-\alpha_5)}}$$

$$k^2_{\alpha_2} = -\frac{1}{x_{\alpha_2} x_{(-\alpha_3)} x_{(-\alpha_4)}} + \frac{1}{x_{\alpha_2} x_{\alpha_3} x_{\alpha_4}} + \frac{1}{x_{\alpha_2} x_{(-\alpha_1)} x_{(-\alpha_4)}} + \frac{1}{x_{\alpha_1} x_{\alpha_2} x_{(-\alpha_4)}} - \frac{1}{x_{\alpha_2} x_{\alpha_3} x_{(-\alpha_5)}} - \frac{1}{x_{\alpha_2} x_{\alpha_3} x_{\alpha_5}}$$

$$k^1_{\alpha_2} = \frac{1}{x_{(-\alpha_3)} x_{(-\alpha_4)}} + \frac{1}{x_{(-\alpha_2)} x_{(-\alpha_5)}} - \frac{1}{x_{\alpha_1} x_{(-\alpha_4)}} + \frac{1}{x_{\alpha_1} x_{\alpha_2}} + \frac{1}{x_{\alpha_3} x_{\alpha_5}}$$

Generating $D^F_{I_2(5)}$

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Set $s_j s_k = s_{jk}$. Let $I_2(m) = I_2(5)$. Then

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Generating $D^F_{I_2(5)}$ - Proof

To begin the proof, we will write out explicitly the products of X_1 and X_2 ending in X_2.

$$X_2 = \frac{1}{x_{\alpha_2}}(\mathbf{1} - \delta_2)$$

$$X_1X_2 = \frac{1}{x_{\alpha_1}x_{\alpha_2}}(\mathbf{1} - \delta_2) + \frac{1}{x_{\alpha_1}x_{(-\alpha_4)}}(\delta_{12} - \delta_1)$$

$$X_2X_1X_2 = X_2(X_1X_2) =$$
$$(\frac{1}{x_{\alpha_1}x^2_{\alpha_2}} + \frac{1}{x_{\alpha_2}x_{(-\alpha_2)}x_{(-\alpha_5)}})(\mathbf{1} - \delta_2) + \frac{1}{x_{\alpha_1}x_{\alpha_2}x_{(-\alpha_4)}}(\delta_{12} - \delta_1) + \frac{1}{x_{\alpha_2}x_{\alpha_3}x_{(-\alpha_5)}}(\delta_{21} - \delta_{212})$$

$$X_2X_1X_2X_1X_2 = (X_2X_1)(X_2X_1X_2) =$$
$$+(\frac{1}{x^2_{\alpha_1}x^3_{\alpha_2}} + \frac{1}{x_{\alpha_1}x^2_{\alpha_2}x_{(-\alpha_2)}x_{(-\alpha_5)}} + \frac{1}{x_{\alpha_1}x^2_{\alpha_2}x_{(-\alpha_1)}x_{(-\alpha_4)}} + \frac{1}{x_{\alpha_2}x^2_{(-\alpha_2)}x^2_{(-\alpha_5)}} + \frac{1}{x_{\alpha_1}x^2_{\alpha_2}x_{(-\alpha_2)}x_{(-\alpha_5)}} +$$
$$\frac{1}{x_{\alpha_2}x_{\alpha_3}x_{\alpha_5}x_{(-\alpha_2)}x_{(-\alpha_5)}})(\mathbf{1} - \delta_2)+$$
$$+(\frac{1}{x_{\alpha_1}x_{\alpha_2}x_{(-\alpha_1)}x^2_{(-\alpha_4)}} + \frac{1}{x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}x_{\alpha_4}x_{(-\alpha_4)}} + \frac{1}{x^2_{\alpha_1}x^2_{\alpha_2}x_{(-\alpha_4)}} + \frac{1}{x_{\alpha_1}x_{\alpha_2}x_{(-\alpha_2)}x_{(-\alpha_4)}x_{(-\alpha_5)}})(\delta_{12} -$$
$$\delta_1)+$$
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$$\delta_{212})+$$
$$+\frac{1}{x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}x_{(-\alpha_4)}x_{(-\alpha_5)}}(\delta_{1212} - \delta_{121} + \delta_{2121} - \delta_{21212})$$

Let $f : Q_{I_2(5)} \mapsto Q_{I_2(5)}$ be a function mapping a polynomial $q(\alpha_1, ..., \alpha_5, -\alpha_1, ..., -\alpha_5, \delta_1, \delta_2)$ to $q(-\alpha_1, ..., -\alpha_5, \alpha_1, ..., \alpha_5, \delta_2, \delta_1)$. If $s_i(q(\alpha, \delta_w)) = q(s_i(\alpha), \delta_w)$, then one can show, for $i = 1, 2, 3, 5$, that

$$\underbrace{X_1 \dots}_{\text{i elements}} = s_{12121}(f(\underbrace{X_2 \dots}_{\text{i elements}}))$$

The strategy of this proof is to show that Equation 1 holds by a "left side equals right side" argument:

$$X_2X_1X_2X_1X_2 = +k^3_{\alpha_2}(X_2X_1X_2) + k^2_{\alpha_2}(X_1X_2) + k^1_{\alpha_2}(X_2) + \frac{1}{x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}x_{(-\alpha_4)}x_{(-\alpha_5)}}(-\delta_{12121} + \delta_{1212} + \delta_{2121} - \delta_{212} - \delta_{121} + \delta_{12} + \delta_{21} - \delta_1 - \delta_2 + \mathbf{1})$$

Equation 2 follows by applying $s_{12121} \circ f$ to Equation 1:

$$X_1X_2X_1X_2X_1 = +s_{12121}(k^3_{-\alpha_2})(X_1X_2X_1) + s_{12121}(k^2_{-\alpha_2})(X_2X_1) + s_{12121}(k^1_{-\alpha_2})(X_1) + \frac{1}{x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}x_{(-\alpha_4)}x_{(-\alpha_5)}}(-\delta_{12121} + \delta_{1212} + \delta_{2121} - \delta_{121} + -\delta_{212} + \delta_{21} + \delta_{12} - \delta_1 - \delta_2 + \mathbf{1})$$

By subtracting Equation 2 from Equation 1, the result follows.

Let $f : Q_{I_2(5)} \mapsto Q_{I_2(5)}$ be a function mapping a polynomial $q(\alpha_1, ..., \alpha_5, -\alpha_1, ..., -\alpha_5, \delta_1, \delta_2)$ to $q(-\alpha_1, ..., -\alpha_5, \alpha_1, ..., \alpha_5, \delta_2, \delta_1)$. If $s_i(q(\alpha, \delta_w)) = q(s_i(\alpha), \delta_w)$, then one can show, for $i = 1, 2, 3, 5$, that

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$$X_2X_1X_2X_1X_2 = +k^3_{\alpha_2}(X_2X_1X_2) + k^2_{\alpha_2}(X_1X_2) + k^1_{\alpha_2}(X_2) + \frac{1}{x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}x_{(-\alpha_4)}x_{(-\alpha_5)}}(-\delta_{12121} + \delta_{1212} + \delta_{2121} - \delta_{212} - \delta_{121} + \delta_{12} + \delta_{21} - \delta_1 - \delta_2 + \mathbf{1})$$

Equation 2 follows by applying $s_{12121} \circ f$ to Equation 1:

$$X_1X_2X_1X_2X_1 = +s_{12121}(k^3_{-\alpha_2})(X_1X_2X_1) + s_{12121}(k^2_{-\alpha_2})(X_2X_1) + s_{12121}(k^1_{-\alpha_2})(X_1) + \frac{1}{x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}x_{(-\alpha_4)}x_{(-\alpha_5)}}(-\delta_{12121} + \delta_{1212} + \delta_{2121} - \delta_{121} + -\delta_{212} + \delta_{21} + \delta_{12} - \delta_1 - \delta_2 + \mathbf{1})$$

By subtracting Equation 2 from Equation 1, the result follows.

Let $f : Q_{I_2(5)} \mapsto Q_{I_2(5)}$ be a function mapping a polynomial $q(\alpha_1, ..., \alpha_5, -\alpha_1, ..., -\alpha_5, \delta_1, \delta_2)$ to $q(-\alpha_1, ..., -\alpha_5, \alpha_1, ..., \alpha_5, \delta_2, \delta_1)$. If $s_i(q(\alpha, \delta_w)) = q(s_i(\alpha), \delta_w)$, then one can show, for $i = 1, 2, 3, 5$, that

$$\underbrace{X_1 \dots}_{\text{i elements}} = s_{12121}(f(\underbrace{X_2 \dots}_{\text{i elements}}))$$

The strategy of this proof is to show that Equation 1 holds by a "left side equals right side" argument:

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By subtracting Equation 2 from Equation 1, the result follows.

Example

Let (F, R) be a formal group law in variables x_{α_i}, x_{α_j}, $i, j = 1, \ldots, 5$, such that $x_{-\alpha_i} = -x_{\alpha_i}$. Then

$$k^3_{\alpha_1} = k^3_{\alpha_2} = \frac{x_{\alpha_2}x_{\alpha_3}x_{\alpha_5} + x_{\alpha_3}x_{\alpha_4}x_{\alpha_5} + x_{\alpha_1}x_{\alpha_2}x_{\alpha_5} + x_{\alpha_1}x_{\alpha_2}x_{\alpha_4} + x_{\alpha_1}x_{\alpha_3}x_{\alpha_4}}{x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}x_{\alpha_4}x_{\alpha_5}}$$

$$k^2_{\alpha_1} = k^2_{\alpha_2} = 0$$

$$k^1_{\alpha_1} = k^1_{\alpha_2} = -\frac{(x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5})}{x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}x_{\alpha_4}x_{\alpha_5}}$$

Suppose $x_{\alpha_i} = \alpha_i$ for each $i = 1, \ldots, 5$. Then

$$k^3_{\alpha_1} = k^2_{\alpha_1} = k^1_{\alpha_1} = 0 \quad \text{and}$$

$$X_1X_2X_1X_2X_1 = X_2X_1X_2X_1X_2$$

Special Cases

Example

Let (F, R) be a formal group law in variables x_{α_i}, x_{α_j}, $i, j = 1, ..., 5$, such that $x_{-\alpha_i} = -x_{\alpha_i}$. Then

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Suppose $x_{\alpha_i} = \alpha_i$ for each $i = 1, ..., 5$. Then

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Let (F, R) be a formal group law in variables x_{α_i}, x_{α_j}, $i, j = 1, \ldots, 5$, such that $x_{-\alpha_i} = \frac{x_{\alpha_i}}{Ax_{\alpha_i} - 1}$ for some $0 \neq A \in R$. Then

$$k^3_{\alpha_1} = k^3_{\alpha_2} = \frac{1}{x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}x_{\alpha_4}x_{\alpha_5}}(x_{\alpha_3}x_{\alpha_4}x_{\alpha_5}(1 - Ax_{\alpha_2}) + x_{\alpha_1}x_{\alpha_2}x_{\alpha_4}(1 - Ax_{\alpha_5}) + x_{\alpha_1}x_{\alpha_3}x_{\alpha_5}(1 - Ax_{\alpha_4}) + x_{\alpha_1}x_{\alpha_2}x_{\alpha_5}(1 - Ax_{\alpha_3}) + x_{\alpha_2}x_{\alpha_3}x_{\alpha_5}(1 - Ax_{\alpha_1}) + 2A^2x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}x_{\alpha_4}x_{\alpha_5})$$

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In General

Lemma

Let $I_2(m)$ be the dihedral group of order m. Let α_1 and α_2 be the simple roots of the corresponding root system. Then the formal affine Demazure Algebra of $I_2(m)$ is generated over Q_F modulo the following relations

- $X_i^2 = KX_i$ *for* $i = 1, ..., m$, *where* $K = \frac{1}{x_i} + \frac{1}{x_{-i}}$
- $\underbrace{X_1 X_2 ... X_1}_{m\text{-elements}} - \underbrace{X_2 X_1 ... X_2}_{m\text{-elements}} = \sum_{i=1}^{m-2} \kappa_{\alpha_1}^i \underbrace{X_j X_k ... X_1}_{i\text{-elements}} - \kappa_{\alpha_2}^i \underbrace{X_k X_j ... X_2}_{i\text{-elements}},$

  $\kappa_{\alpha_1}^i, \kappa_{\alpha_2}^i \in Q_F$.

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Theorem

Let $I_2(m)$ be the dihedral group of order m. Let $y_{\alpha_i} = \frac{1}{x_{\alpha_i}}$, $i = 1, 2$. We have an explicit formula for the coefficient $\kappa_{\alpha_1}^{(m-2)}$ from the previous lemma:

$$\kappa_{\alpha_1}^{(m-2)} = y_{\alpha_1} y_{\alpha_2} + s_{\alpha_1}(y_{\alpha_1} y_{\alpha_2}) + \ldots + \underbrace{s_{\alpha_1} s_{\alpha_2} \ldots}_{m-2}(y_{\alpha_1} y_{\alpha_2}) - y_{\alpha_2} s_{\alpha_1}^{(m-2)}(y_{\alpha_2}).$$

$\kappa_{\alpha_2}^{(m-2)}$ is obtained from $\kappa_{\alpha_1}^{(m-2)}$ by switching α_1 and α_2.

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Let $I_2(m)$ be the dihedral group of order m. Let $y_{\alpha_i} = \frac{1}{x_{\alpha_i}}$, $i = 1, 2$. We have an explicit formula for the coefficient $\kappa_{\alpha_1}^{(m-2)}$ from the previous lemma:

$$\kappa_{\alpha_1}^{(m-2)} = y_{\alpha_1} y_{\alpha_2} + s_{\alpha_1}(y_{\alpha_1} y_{\alpha_2}) + \ldots + \underbrace{s_{\alpha_1} s_{\alpha_2} \ldots}_{m-2}(y_{\alpha_1} y_{\alpha_2}) - y_{\alpha_2} s_{\alpha_1}^{(m-2)}(y_{\alpha_2}).$$

$\kappa_{\alpha_2}^{(m-2)}$ is obtained from $\kappa_{\alpha_1}^{(m-2)}$ by switching α_1 and α_2.

In General

Theorem

Let $I_2(m)$ be the dihedral group of order m. Let $y_{\alpha_i} = \frac{1}{x_{\alpha_i}}$, $i = 1, 2$. We have an explicit formula for the coefficient $\kappa_{\alpha_1}^{(m-2)}$ from the previous lemma:

$$\kappa_{\alpha_1}^{(m-2)} = y_{\alpha_1} y_{\alpha_2} + s_{\alpha_1}(y_{\alpha_1} y_{\alpha_2}) + \ldots + \underbrace{s_{\alpha_1} s_{\alpha_2} \ldots}_{m-2}(y_{\alpha_1} y_{\alpha_2}) - y_{\alpha_2} s_{\alpha_1}^{(m-2)}(y_{\alpha_2}).$$

$\kappa_{\alpha_2}^{(m-2)}$ is obtained from $\kappa_{\alpha_1}^{(m-2)}$ by switching α_1 and α_2.