Motivic Segre classes of Schubert cells and the connective formal group law (Raj Gandhi)

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Divided difference operators

Define $R := \mathbb{Z}[x_1, \ldots, x_n]$. Let s_i be the transposition in S_n that swaps i and $i+1$. This defines an action of S_n on R, where s_i swaps x_i and x_{i+1}.

Definition (Demazure 1973, 1974)

Consider the $\mathbb{Z}$-linear operators on R, one for each $i = 1, \ldots, n-1$:

$$\partial_i(f) := \frac{f - s_i(f)}{x_i - x_{i+1}}, \quad f \in R.$$

The ∂_i are called **divided difference operators**.

For $w = s_{i_1} \circ \cdots s_{i_k}$ reduced, define $\partial_w := \partial_{s_{i_1}} \circ \cdots \circ \partial_{s_{i_k}}$. The operator ∂_w does not depend on the choice of reduced expression for w.

Example

$$\partial_2(x_1x_3) = \frac{x_1x_3 - s_2(x_1x_3)}{x_2 - x_3} = \frac{x_1x_3 - x_1x_2}{x_2 - x_3} = -x_1 \in R.$$

S_n-actions

Let Λ_k^n be the set of 01 sequences with k 1's and $n-k$ 0's. The swap s_i acts on Λ_k^n by swapping the i-th and $(i+1)$-th entries of a sequence. Define the word $\omega := 1^k 0^{n-k}$.

Example

The sequence $1001110 \in \Lambda_4^7$. We have $s_1(1001110) = 0101110$.

Consider the ring $\widetilde{R} := \bigoplus_{\Lambda_k^n} R = \bigoplus_{\Lambda_k^n} \mathbb{Z}[x_1, \dots, x_n]$.

The transposition s_i acts on $\widetilde{R}$ by $s_i((f_\lambda)_{\lambda \in \Lambda_k^n}) := (s_i(f_\lambda))_{s_i(\lambda) \in \Lambda_k^n}$.

Example

Consider $(f_{110}, f_{101}, f_{011}) = (x_1x_2, x_2^2, x_1x_3^4) \in \widetilde{R}$, indexed by Λ_2^3. Then

$$s_1(x_1x_2, x_2^2, x_1x_3^4) = (s_1(x_1x_2), s_1(x_1x_3^4), s_1(x_2^2)) = (x_1x_2, x_2x_3^4, x_1^2).$$

GKM conditions and Schubert classes

Definition (Goresky-Kottwitz-MacPherson 1996)

An element $(f_\lambda)_{\lambda \in \Lambda_k^n} \in \widetilde{R}$ is called **GKM** if:

whenever $\lambda = (i,j)(\lambda')$, the difference $f_\lambda - f_{\lambda'}$ is divisible by $x_i - x_j$ in R.

Example

The sequences $(1,1,1)$ and $(0,0,(x_1-x_2)(x_1-x_3)(x_2-x_3))$ in $\widetilde{R}$ indexed by $\Lambda_1^3 = \{(f_{100}, f_{010}, f_{001})\}$ are GKM.

Definition (Schubert classes)

Fix Λ_k^n. Define in $\widetilde{R}$, an element $S_\omega|_\lambda := \begin{cases} \prod_{i>j:\lambda_i<\lambda_j} x_j - x_i, & \text{if } \lambda = \omega; \\ 0, & \text{otherwise.} \end{cases}$

The other S_λ are defined by the rule $S_{w^{-1}(\omega)} := \partial_w(S_\omega)$.

The S_λ are GKM and called **Schubert classes**.

Localization package

Theorem

Let X be a smooth complex algebraic variety that has an algebraic action of a complex torus $T := (\mathbb{C}^\times)^n$, and assume this action has finitely many fixed points F. The natural ring homomorphisms

$$H_T(X) \to \bigoplus_{f \in F} H_T(\mathrm{pt}) \simeq \bigoplus_{f \in F} \mathbb{Z}[x_1, \ldots, x_n],$$

induced by the inclusions $\{\text{fixed point}\} \hookrightarrow X$, is injective.

Example

When X is the Grassmannian $\mathrm{Gr}(k, n)$, whose points are the k-dimensional subspaces of $\mathbb{C}^n$, the image of the localization map above is determined by the GKM conditions.

Multiplying Schubert classes

Definition (Schubert basis)

The $\mathbb{Z}[x_1, \ldots, x_n]$-subalgebra of $\widetilde{R}$ generated by $\{S_\lambda\}_{\lambda \in \Lambda_k^n}$ is $H_T(\mathrm{Gr}(k, n))$. The S_λ form $\mathbb{Z}[x_1, \ldots, x_n]$-basis for the subalgebra: the **Schubert basis**.

Let us run an example for Λ_1^2. Recall the operator

$$\partial_1(f) := \frac{f - s_1(f)}{x_1 - x_2}.$$

We have

$$S_{10} = [0, x_1 - x_2]\,; \qquad S_{01} = \partial_1(S_{10}) = [1, 1]\,.$$

Let us compute all products and express them in terms of the S_λ:

$$S_{10}^2 = (x_1 - x_2)S_{10}; \quad S_{10} \cdot S_{01} = S_{10}; \quad S_{01}^2 = S_{01}.$$

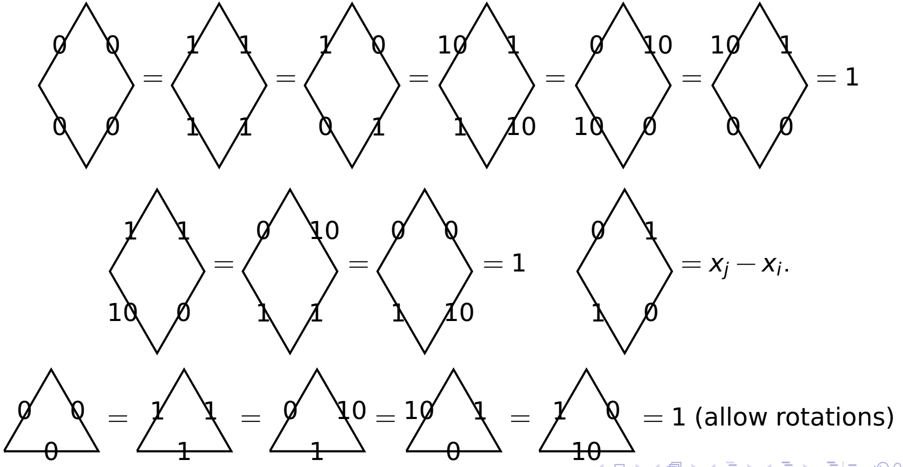
The structure constants lie in $\mathbb{N}[x_1 - x_2]$.

Question

Is there a combinatorial formula for the structure constants in S_λ basis?

Knutson-Tao puzzles

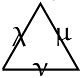
Consider the following **puzzle pieces**, equipped with a function from $\{1, 2, 3, \dots\}^2$ to $\mathbb{Z}[x_1, x_2, \dots]$ called its **fugacity**.



Knutson-Tao puzzles

A **Knutson-Tao puzzle** is a triangle with side labels λ, μ, ν in Λ_k^n that is tiled by the puzzle pieces.

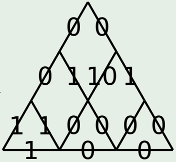
The **fugacity** of a puzzle is the product of fugacities of its tiles. The fugacity of a rhombus tile is $x_i - x_j$, where i is the i-th NE-to-SW diagonal, and j is the j-th NW-to-SE diagonal in the puzzle.



$:=$ sum of puzzle fugacities over all puzzles with λ, μ, ν boundary.

Example

For $\lambda = 100$ (left), $\mu = 010$ (right), $\nu = 100$ (bottom):

fug 

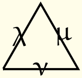
$= (x_1 - x_2) \cdot 1 \cdot 1 = x_1 - x_2.$

Knutson-Tao puzzles

Theorem (Knutson-Tao 2003)

For any $\lambda, \mu \in \Lambda_k^n$, the product $S_\lambda \cdot S_\mu$ is

$$S_\lambda \cdot S_\mu = \sum_\nu$$



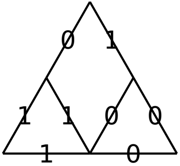
$$S_\nu .$$

Thus the structure constants lie in $\mathbb{N}[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]$.

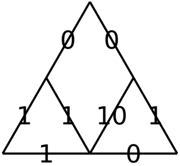
Recall our computation in a previous example:

$$S_{10}^2 = (x_1 - x_2)S_{10}; \quad S_{10} \cdot S_{01} = S_{10}; \quad S_{01}^2 = S_{01}.$$

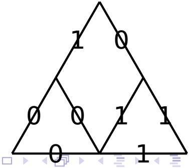
We compute



$= x_1 - x_2$



$= 1$



$= 1$

Positive formulas

Question

What is a positive formula?

Example

Say I have a basis $B_1, \ldots, B_n$, and the structure constants for this basis live in $\mathbb{N}$. The structure constants are **positive** because $\mathbb{N}$ is a monoid and $\mathbb{N} \cap (-\mathbb{N}) = (0)$.

Definition (Knutson–Zinn-Justin 2021)

A **positivity monoid** is a monoid M such that $M \cap (-M) = (0)$. If the structure constants for a basis live in a positivity monoid, then the structure constants are **positive**.

Example

$\mathbb{N}[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]$ is a positivity monoid.

Motivic Segre classes of Schubert cells

Define $R := \mathbb{Z}[e^{\pm x_1}, \ldots, e^{\pm x_n}, q^{\pm 1}]$. Define an action of S_n on R, where s_i swaps e^{x_i} and $e^{x_{i+1}}$ and fixes q^2. Define the ring $\widetilde{R} := \oplus_{\lambda \in \Lambda_k^n} \mathrm{Frac}(R)$.

Definition

Consider the $\mathbb{Z}[q^{\pm 1}]$-linear operators on R, one for each $i = 1, \ldots, n-1$:

$$\partial_i := \frac{1 - q^2}{1 - e^{x_{i+1} - x_i}} + \frac{1 - q^2 e^{x_i - x_{i+1}}}{1 - e^{x_i - x_{i+1}}} s_i.$$

The ∂_i are called **Demazure-Lusztig operators**.

Define in $\widetilde{R}$ an element $S_\omega|_\lambda := \begin{cases} \prod_{i>j : \lambda_i < \lambda_j} \frac{1 - e^{x_j - x_i}}{1 - q^2 e^{x_j - x_i}}, & \text{if } \lambda = \omega; \\ 0, & \text{otherwise.} \end{cases}$

The other S_λ are defined by the rule $S_{w^{-1}(\omega)} := \partial_w(S_\omega)$.

The S_λ are called **motivic Segre classes of Schubert cells**.

There is a positive puzzle formula for the structure constants for S_λ in terms of Knutson-Tao puzzles [Knutson–Zinn-Justin 2021].

Deforming the motivic Segre classes

Let's strategically add a parameter β to the previous recursion:

$$\partial_i := \frac{\beta(1-q^2)}{1-e^{x_{i+1}-x_i}} + \frac{\beta(1-q^2)+q^2(1-e^{x_i-x_{i+1}})}{1-e^{x_i-x_{i+1}}} s_i.$$

$$S_\omega|_\lambda := \begin{cases} \prod_{i>j:\lambda_i<\lambda_j} \frac{1-e^{x_i-x_j}}{\beta(1-q^2)+q^2(1-e^{x_i-x_j})}, & \text{if } \lambda=\omega; \\ 0, & \text{otherwise.} \end{cases}$$

$$S_{w^{-1}(\omega)} := \partial_w(S_\omega)$$

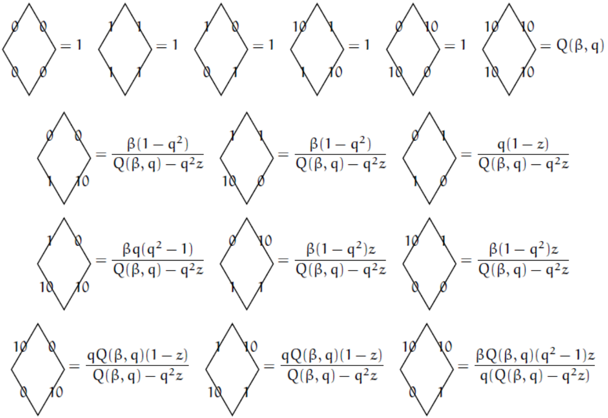
Lemma

$\partial_w := \partial_{i_1} \circ \cdots \circ \partial_{i_k}$ is independent of the reduced expression $w = s_{i_1} \cdots s_{i_k}$:

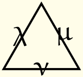
1. $\partial_i \circ \partial_{i+1} \circ \partial_i = \partial_{i+1} \circ \partial_i \circ \partial_{i+1}$ for $i = 1, \ldots, n-2$.
2. $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ for all $|i-j| > 1$.

The $\beta = 1$ specialization recovers the motivic Segre classes S_λ.

The $\beta = 0$ 'limit' recovers the 'Segre-Schwarts-MacPherson classes'.



Theorem (G. 2026+)

$$(q^{\mathrm{length}(\lambda)} S_\lambda) \cdot (q^{\mathrm{length}(\mu)} S_\mu) = \sum_\nu$$  $$(q^{\mathrm{length}(\nu)} S_\nu)$$

Positivity

Define $Q(\beta) := q^2 + \beta - q^2\beta$.

Consider the submonoid M of $\mathrm{Frac}(\mathbb{Z}[\beta][e^{\pm x_1}, \ldots, e^{\pm x_n}, q^{\pm 1}])$, defined as the set of sums of products of the factors over all $1 \leqslant i < j \leqslant n$:

$$-q^{\pm} \qquad Q(\beta) \qquad e^{x_j - x_i} \qquad \frac{\beta(1-q^2)}{\beta(1-q^2)+q^2(1-e^{x_j-x_i})} \qquad -\frac{1-e^{x_j-x_i}}{\beta(1-q^2)+q^2(1-e^{x_j-x_i})}.$$

Then M is a positivity monoid.

As the structure constants in the S_λ basis live in M, it is in this sense that our puzzle formula is positive.

The formal group algebra $\mathcal{S}$

Consider the connective formal group law $F_C = x + y - \beta xy$ over $\mathbb{Z}[\beta]$. Let T be a max. torus in $\mathrm{GL}_n(\mathbb{C})$, with weight lattice $\Lambda := \mathrm{Hom}(T, \mathbb{C}^\times)$. Form the formal group algebra $\mathcal{S} := \mathbb{Z}[\beta][\![x_\lambda]\!]_{\lambda\in\Lambda}/\mathcal{J}_{F_C}$, where $\mathcal{J}_{F_C}$ is the closure of the ideal in $\mathbb{Z}[\beta][\![x_\lambda]\!]_{\lambda\in\Lambda}$ generated by the relations

$$x_0 \quad \text{and} \quad x_{\lambda_1+\lambda_2} = F_C(x_{\lambda_1}, x_{\lambda_2}), \quad \lambda_1, \lambda_2 \in \Lambda.$$

Note: $K_T(\mathrm{pt}) \simeq \mathbb{Z}[e^{\pm y_1}, \dots, e^{\pm y_n}]$ and $H_T(\mathrm{pt}) \simeq \mathbb{Z}[y_1, \dots, y_n]$.

Lemma

There are ring isomorphisms

$$\mathcal{S} \hookrightarrow \mathcal{S}_\beta \xrightarrow{\simeq} \widehat{K}_T(\mathrm{pt}) \otimes \mathbb{Z}[\beta, \beta^{-1}], \quad x_\lambda \mapsto \beta^{-1}(1 - e^\lambda),$$

$$\mathcal{S}/(\beta) \xrightarrow{\simeq} \widehat{H}_T(\mathrm{pt}), \quad x_\lambda \mapsto \lambda.$$

Here, $\widehat{K}_T(\mathrm{pt})$ (resp. $\widehat{H}_T(\mathrm{pt})$) is the completion of $K_T(\mathrm{pt})$ (resp. $H_T(\mathrm{pt})$) at the ideal generated by the elements $1 - e^\lambda$ (resp. λ) over all $\lambda \in \Lambda$.

The connective motivic Segre classes

Consider the divided difference operator and classes in $\oplus_{\lambda\in\Lambda_k^n}\mathrm{Frac}(\mathcal{S}[\![q]\!])$:

$$\partial_i := \frac{1-q^2}{x_{-\alpha_i}} + \frac{1-q^2+q^2x_{\alpha_i}}{x_{\alpha_i}}s_i.$$

$$S_\omega|_\lambda := \begin{cases} \prod_{i>j:\lambda_i<\lambda_j} \frac{x_{x_i-x_j}}{1-q^2+q^2x_{x_i-x_j}}, & \text{if } \lambda=\omega; \\ 0, & \text{otherwise.} \end{cases}$$

$$S_{w^{-1}(\omega)} := \partial_w(S_\omega).$$

Lemma

When β is *nonzero*, the classes S_λ are identified with the β-deformed classes defined earlier, via $\mathcal{S}[\![q]\!] \hookrightarrow \widehat{K}_T(\mathrm{pt})[\![q]\!]$.
When $\beta=0$, the classes S_λ are identified with the *Segre-Schwartz-MacPherson classes of Schubert cells* in $\mathrm{Gr}(k,n)$.

The GKM condition

Clear the denominators in the S_λ to define classes St_λ:

$$\mathrm{St}_\omega := \left(\prod_{i>j:\omega_i<\omega_j} (1 - q^2 + q^2 x_{x_i - x_j}) \right) S_\omega; \quad \mathrm{St}_{w^{-1}(\omega)} := \partial_w(\mathrm{St}_\omega).$$

Proposition (G. 2026+)

The elements St_λ satisfy:

$$\text{when } \lambda = (i,j)(\lambda'), \text{ the difference } \mathrm{St}_\lambda - \mathrm{St}_{\lambda'} \text{ is divisible by } x_{x_i - x_j}.$$

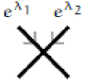
We call the St_λ the **connective motivic Chern classes**. They live in a quotient of the equivariant algebraic cobordism ring of $T^*(\mathrm{Gr}(k,n))$.

Question (Open)

Do the St_λ come from canonical elements in the $(T \times \mathbb{C}^\times)$-equivariant connective K-ring of $T^*(\mathrm{Gr}(k,n))$?

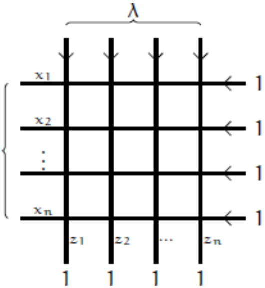
Rational function representatives for deformed classes

$$\widehat{R}(\beta, e^{\lambda})_K :=$$



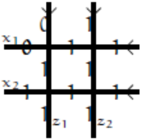
$$= \begin{array}{c} \\ 1\vee 1 \\ 1\vee 0 \\ 0\vee 1 \\ 0\vee 0 \end{array} \begin{array}{c} \begin{array}{cccc} 1\wedge 1 & 1\wedge 0 & 0\wedge 1 & 0\wedge 0 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\beta(1-q^2)e^{\lambda}}{Q(\beta)-q^2e^{\lambda}} & \frac{qQ(\beta)(1-e^{\lambda})}{Q(\beta)-q^2e^{\lambda}} & 0 \\ 0 & \frac{q(1-e^{\lambda})}{Q(\beta)-q^2e^{\lambda}} & \frac{\beta(1-q^2)}{Q(\beta)-q^2e^{\lambda}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

$S_\lambda := \omega$

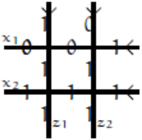


"Sum over all possible grids, and add the fugacities together"

Rational function representatives for deformed classes

$S_{01} =$ 

$= \dfrac{\beta(1-q^2)}{Q(\beta) - q^2(x_1/z_1)}$

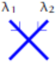
$S_{10} =$ 

$= \dfrac{q(1-x_1/z_1)}{Q(\beta) - q^2(x_1/z_1)} \cdot \dfrac{\beta(1-q^2)}{Q(\beta) - q^2(x_1/z_2)}$

The rational functions S_λ represent the homogenizations $q^{\mathrm{length}(\lambda)} S_\lambda$ of the deformed classes S_λ defined earlier.

Proof of puzzle rule: rational function R-matrix

The rational functions S_λ can also be defined using the following matrix entries, with $x_\lambda = \beta^{-1}(1 - e^\lambda)$ and $y_\lambda = \beta(1-q^2) + q^2(1 - e^\lambda)$.

$R_{bb}(\beta, x_\lambda) =$ 

$=$

	1/\1	1/\0	1/\10	0/\1	0/\0	0/\10	10/\1	10/\0	10/\10
1\/1	1	0	0	0	0	0	0	0	0
1\/0	0	$\frac{(1-q^2)(1-\beta x_\lambda)}{y_\lambda}$	0	$\frac{Q(\beta) q x_\lambda}{y_\lambda}$	0	0	0	0	0
1\/10	0	0	$\frac{1-q^2}{y_\lambda}$	0	0	0	$\frac{Q(\beta) q x_\lambda}{y_\lambda}$	0	0
0\/1	0	$\frac{q x_\lambda}{y_\lambda}$	0	$\frac{1-q^2}{y_\lambda}$	0	0	0	0	0
0\/0	0	0	0	0	1	0	0	0	0
0\/10	0	0	0	0	0	$\frac{1-q^2}{y_\lambda}$	0	$\frac{q x_\lambda}{y_\lambda}$	0
10\/1	0	0	$\frac{q x_\lambda}{y_\lambda}$	0	0	0	$\frac{(1-q^2)(1-\beta x_\lambda)}{y_\lambda}$	0	0
10\/0	0	0	0	0	0	$\frac{Q(\beta) q x_\lambda}{y_\lambda}$	0	$\frac{(1-q^2)(1-\beta x_\lambda)}{y_\lambda}$	0
10\/10	0	0	0	0	0	0	0	0	1

Puzzle fugacities encoded in a matrix

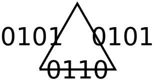
$R_{gr}(\beta, x_\lambda) =$ 

$=$

	1/\1	1/\0	1/\10	0/\1	0/\0	0/\10	10/\1	10/\0	10/\10
1\/1	1	0	0	0	0	$\frac{(1-q^2)(1-\beta x_\lambda)}{y_\lambda}$	0	0	0
1\/0	0	0	0	$\frac{q x_\lambda}{y_\lambda}$	0	0	0	0	0
1\/10	0	0	0	0	$\frac{1-q^2}{y_\lambda}$	0	1	0	0
0\/1	0	1	0	0	0	0	0	0	$\frac{Q(\beta)(q^2-1)(1-\beta x_\lambda)}{q y_\lambda}$
0\/0	0	0	0	0	1	0	$\frac{(1-q^2)(1-\beta x_\lambda)}{y_\lambda}$	0	0
0\/10	0	0	0	0	0	0	0	$\frac{Q(\beta) q x_\lambda}{y_\lambda}$	0
10\/1	0	0	$\frac{Q(\beta) q x_\lambda}{y_\lambda}$	0	0	0	0	0	0
10\/0	$\frac{1-q^2}{y_\lambda}$	0	0	0	0	1	0	0	0
10\/10	0	$\frac{q(q^2-1)}{y_\lambda}$	0	0	0	0	0	0	$Q(\beta)$

Proof of the puzzle rule

There are also $R_{r,r}$, $R_{g,g}$, $R_{g,b}$, and $R_{b,r}$ matrices.

The following diagram equals 

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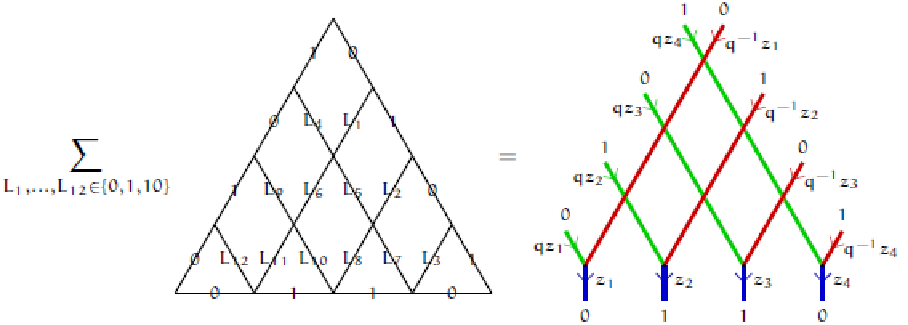
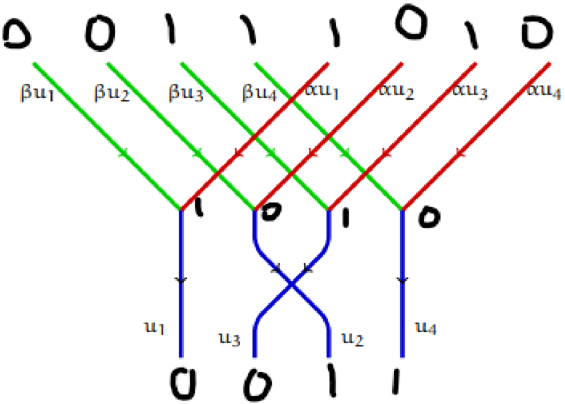


Proof of the puzzle rule

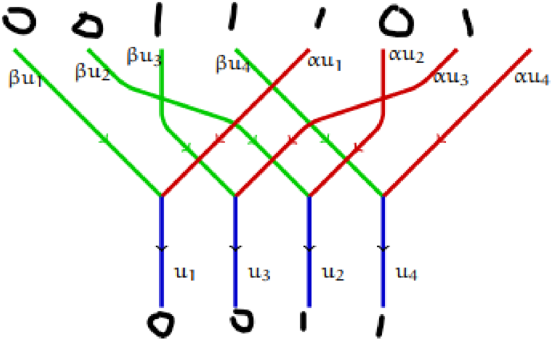
The following diagram computes $\underset{1010}{{}^{0011}\triangle^{1010}}\ S_{1010}|_{0101}$.



Removing 1010 in the center, it computes $\sum_\nu \underset{\nu}{{}^{0011}\triangle^{1010}}\ S_\nu|_{0101}$.

Proof of the puzzle rule

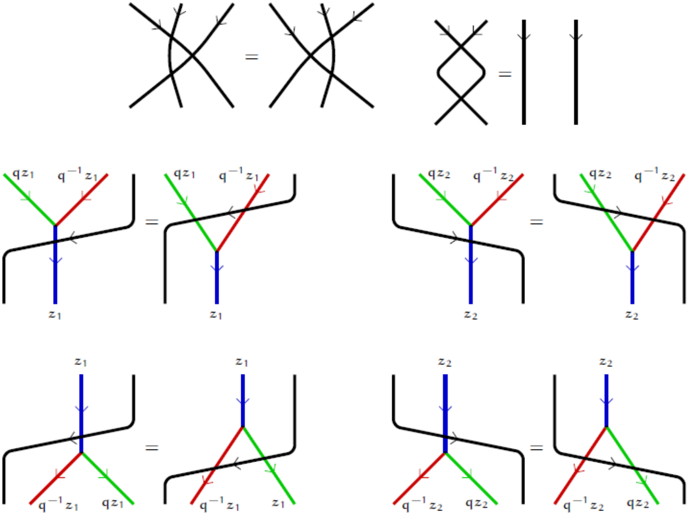
The following diagram computes $S_{0011}|_{0101} \cdot S_{1010}|_{0101}$ (I am sweeping details under the rug!) Note: red and green matrices “equal" blue matrix (almost).



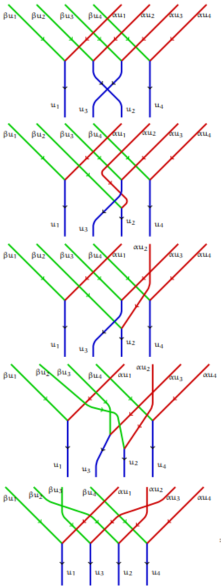
Must prove that this diagram equals previous one! Equality of formula at all restrictions implies equality of classes.

Proof of the puzzle rule

The following hold!



Proof of the puzzle rule



A word about the R-matrices

Question

What exactly *are* the matrices defined earlier?

Is there a quantum group attached to them?

Theorem (G. 2026+)

The matrices used to prove the puzzle formula arise as intertwiners for representations of the **multi-parameter quantum group** of type $\widehat{\mathfrak{a}}_2$.

Set $\mathbf{q} := (q_{i,j})_{i,j=1,\ldots,n}$ with $q_{i,j}q_{j,i}q_{i,i} = 1$ for $i \neq j$, and $\mathbb{K} = \mathbb{Q}(\mathbf{q})$. The **multi-parameter quantum group** $U_{\mathbf{q}}(\widehat{a}_n)$ is the associative unital algebra over $\mathbb{K}$ generated by elements E_i, F_i, $K_i^{(1)}$, $(K_i^{(1)})^{-1}$, $K_i^{(2)}$, $(K_i^{(2)})^{-1}$, $i = 0, 1, \ldots, n-1$, subject to the relations:

- $K_i^{(2)}(K_i^{(2)})^{-1} = (K_i^{(2)})^{-1}K_i^{(2)} = 1, \quad K_i^{(1)}(K_i^{(1)})^{-1} = (K_i^{(1)})^{-1}K_i^{(1)} = 1,$
- $K_i^{(1)}K_j^{(2)} = K_j^{(2)}K_i^{(1)}, \quad K_i^{(1)}K_j^{(1)} = K_j^{(1)}K_i^{(1)}, \quad K_i^{(2)}K_j^{(2)} = K_j^{(2)}K_i^{(2)},$
- $K_i^{(1)}E_j(K_i^{(1)})^{-1} = q_{i,j}E_j, \quad (K_i^{(2)})^{-1}E_jK_i^{(2)} = q_{j,i}^{-1}E_j,$
- $K_i^{(1)}F_j(K_i^{(1)})^{-1} = q_{i,j}^{-1}F_j, \quad (K_i^{(2)})^{-1}F_jK_i^{(2)} = q_{j,i}F_j,$
- $[E_i, F_j] = \delta_{i,j}\frac{q_{i,i}}{q_{i,i}-1}\left(K_i^{(1)} - (K_i^{(2)})^{-1}\right),$
- $E_i^2E_j - q_{i,j}(1+q_{i,i})E_iE_jE_i + \frac{q_{i,j}}{q_{j,i}}E_jE_i^2 = 0, \quad i \neq j,$
- $\frac{q_{i,j}}{q_{j,i}}F_i^2F_j - q_{i,j}(1+q_{i,i})F_iF_jF_i + F_jF_i^2 = 0, \quad i \neq j.$

Future directions

1. Extend our results to $d = 2, 3, 4$-step flag varieties using other multi-parameter quantum groups.
2. Interpret the β-deformed classes as elements in the equivariant *connective* K-ring of $T^*(G/B)$.
3. Extend the definition of the connective motivic Segre classes to define a general theory of stable envelopes for the equivariant connective K-rings of symplectic resolutions.
4. Define an action of the multi-parameter quantum group on the equivariant connective K-rings of Nakajima quiver varieties.

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A note on Chern classes

Remark

The element $1-e^{x_i-x_{i+1}}$ is the first equivariant Chern class (in K-theory) of the homogeneous line bundle $\mathcal{L}_{x_{i+1}-x_i} \to G/B$. Let's replace $1-e^{x_i-x_{i+1}}$ by $c_1(\mathcal{L}_{x_{i+1}-x_i})$ everywhere in the motivic Segre classes.

$$K_T: \qquad \partial_i := \frac{1-q^2}{c_1(\mathcal{L}_{x_i-x_{i+1}})} + \frac{1-q^2(1-c_1(\mathcal{L}_{x_i-x_{i+1}}))}{c_1(\mathcal{L}_{x_{i+1}-x_i})} s_i.$$

$$S_\omega|_\lambda := \begin{cases} \prod_{i>j:\lambda_i<\lambda_j} \frac{c_1(\mathcal{L}_{x_i-x_j})}{1-q^2(1-c_1(\mathcal{L}_{x_i-x_j}))}, & \text{if } \lambda=\omega; \\ 0, & \text{otherwise.} \end{cases}$$

$$S_{w^{-1}(\omega)} := \partial_w(S_\omega).$$

Question

What if we replace c_1 by a Chern class in another cohomology theory?

'Connective' K-theory

An algebraic oriented cohomology theory h^* is a functor:

$$\mathrm{h}^* : \{\text{smooth algebraic varieties}\} \to \{\text{graded, commutative, unital rings}\},$$

that satisfies 'cohomology-type' axioms.

Example

Chow ring theory and K-theory are oriented cohomology theories.

There is an oriented cohomology theory called **connective K-theory**. After a localization, the first equivariant Chern class in connective K-theory sends $\mathcal{L}_{x_{i+1}-x_i}$ to $\beta^{-1}(1-e^{x_i-x_{i+1}})$, where β is a free variable.

Let's replace everything with this new Chern class!

Recall the GKM conditions

Definition

An element $(f_\lambda)_{\lambda\in\Lambda_k^n} \in \bigoplus_{\lambda\in\Lambda_k^n} \mathbb{Z}[x_1,\dots,x_n,\hbar]$ is called **GKM** if:

$$\text{whenever } \lambda = (i,j)(\lambda'), \text{ the difference } f_\lambda - f_{\lambda'} \text{ is divisible by } x_i - x_j.$$

A GKM class $(f_\lambda)_{\lambda\in\Lambda_k^n}$ can be identified with a class in $H_{T\times\mathbb{C}^\times}(T^*\mathrm{Gr}(k,n))$.

Definition

An element $(f_\lambda)_{\lambda\in\Lambda_k^n} \in \bigoplus_{\lambda\in\Lambda_k^n} K_{T\times\mathbb{C}^\times}(\mathrm{pt}) = \bigoplus_{\lambda\in\Lambda_k^n} \mathbb{Z}[e^{\pm x_1},\dots,e^{\pm x_n},q^2]$ is called **GKM** if:

$$\text{whenever } \lambda = (i,j)(\lambda'), \text{ we have } f_\lambda - f_{\lambda'} \text{ is divisible by } 1-e^{x_i-x_j}.$$

A GKM class $(f_\lambda)_{\lambda\in\Lambda_k^n}$ can be identified with a class in $K_{T\times\mathbb{C}^\times}(T^*\mathrm{Gr}(k,n))$.

SSM and motivic Segre classes are quotients of classes that satisfy GKM called 'stable classes'.

Motivic Chern classes of Schubert cells

Define the following elements in $\oplus_{\lambda\in\Lambda_k^n} K_{T\times\mathbb{C}^\times}(\mathrm{pt})$:

$$\mathrm{St}_\omega|_\lambda = \begin{cases} \prod_{i>j:\lambda_i<\lambda_j}(q(1-e^{x_j-x_i})), & \text{if } \lambda=\omega; \\ 0, & \text{otherwise.} \end{cases}, \quad \mathrm{St}_{w^{-1}(\omega)} := \frac{1}{q^{\ell(w)}}\partial_w(\mathrm{St}_\omega).$$

Definition

An element $(f_\lambda)_{\lambda\in\Lambda_k^n} \in \bigoplus_{\lambda\in\Lambda_k^n} K_{T\times\mathbb{C}^\times}(\mathrm{pt}) = \bigoplus_{\lambda\in\Lambda_k^n}\mathbb{Z}[e^{\pm x_1},\dots,e^{\pm x_n},q^{\pm}]$ is called **GKM** if:

whenever $\lambda=(i,j)(\lambda')$, we have $f_\lambda - f_{\lambda'}$ is divisible by $1-e^{x_i-x_j}$.

A GKM class $(f_\lambda)_{\lambda\in\Lambda_k^n}$ can be identified with a class in $K_{T\times\mathbb{C}^\times}(T^*\mathrm{Gr}(k,n))$.

The classes St_ν satisfy the GKM conditions, and are called the **motivic Chern classes of Schubert cells** (a.k.a. stable classes for the positive Weyl alcove closest to the origin, and polarization $T^*\mathrm{Gr}(k,n)$).

A word about equivariant algebraic cobordism

Equivariant algebraic cobordism Ω was invented independently by Krishna and Heller–Malagón-Lopez.

Theorem (Follows from work of Krishna)

The equivariant algebraic cobordism ring $\Omega_{T\times\mathbb{C}^\times}(T^*(\mathrm{Gr}(k,n)))$ is isomorphic to the subring $\mathcal{F}$ of $\oplus_{\lambda\in\Lambda_k^n}\Omega_{T\times\mathbb{C}^\times}(\mathrm{pt})$ whose elements are sequences $(f_\lambda)_{\lambda\in\Lambda_k^n}$ such that $f_\lambda - f_{s_\alpha(\lambda)}$ is divisible by x_α over all roots α.

There is a surjective ring homomorphism $\Omega_{T\times\mathbb{C}^\times}(\mathrm{pt}) \to \mathcal{S}[\![q]\!]$.

Question (Open problem)

Can the elements St_λ be viewed as canonical elements in a quotient of $\Omega_{T\times\mathbb{C}^\times}(T^*(\mathrm{Gr}(k,n)))$?